Math/CS 240, Answers to Selected Homework Problems

1.8.25

A: Let $g : A \to B$ and $f : B \to C$ be injective functions. We wish to show that $f \circ g$ is also injective. Suppose that $(f \circ g)(a_1) = (f \circ g)(a_2)$ for some $a_1, a_2 \in A$. This means that $f(g(a_1)) = f(g(a_2))$. Because f is injective, it follows that $g(a_1) = g(a_2)$; then the injectivity of g implies that $a_1 = a_2$. Thus $a_1 = a_2$ whenever $(f \circ g)(a_1) = (f \circ g)(a_2)$, and this shows $f \circ g$ to be injective.

B: Let $g : A \to B$ and $f : B \to C$ be surjective functions. We wish to show that $f \circ g$ is also surjective. Let c be an arbitrary element of C. Because f is surjective, there exists some $b \in B$ such that $f(b) = c$; since g is surjective, there exists $a \in A$ such that $g(a) = b$. Thus

$$(f \circ g)(a) = f(g(a)) = f(b) = c.$$

We see that all $c \in C$ are in the range of $f \circ g$; thus $f \circ g$ is surjective.

2.2.24

B: We wish to show that $2x^2 + x - 7$ is $\Theta(x^2)$. To do this, it suffices to show that x^2 is $\mathcal{O}(2x^2 + x - 7)$ and that $2x^2 + x - 7$ is $\mathcal{O}(x^2)$. First, for $x \geq 3$, it is true that

$$0 \leq x^2 \leq 2x^2 + x - 7.$$

Thus, if $k = 3$ and $C = 1$, we have $|x^2| \leq C|2x^2 + x - 7|$ for $x \geq k$. This establishes that x^2 is $\mathcal{O}(2x^2 + x - 7)$. Second, for $x \geq 2$,

$$0 \leq 2x^2 + x - 7 \leq 3x^2.$$

So, if $k = 2$ and $C = 3$, then $|2x^2 + x - 7| \leq C|x^2|$ for $x \geq k$. Therefore $2x^2 + x - 7$ is $\mathcal{O}(x^2)$. We conclude that $2x^2 + x - 7$ is $\Theta(x^2)$.

2.4.25

A. $\phi(4) = |\{1, 3\}| = 2$.
B. $\phi(10) = |\{1, 3, 7, 9\}| = 2$.
C. $\phi(13) = |\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}| = 12$.

2.5.28

Let $n = (a_d \cdots a_0)_{10} = a_d 10^d + \cdots a_0 10^0$ be a positive integer of $d + 1$ digits. Let

$$d_{even} = 2 \left\lfloor \frac{d}{2} \right\rfloor,$$

$$d_{odd} \quad = \quad 2\left\lfloor \frac{d-1}{2} \right\rfloor + 1;$$

these are the largest even and odd numbers, respectively, that are less than or equal to d. Let

$$s = (a_0 + a_2 + \cdots + a_{d_{even}}) - (a_1 + a_3 + \cdots a_{d_{odd}}).$$

Because $10 \equiv_{11} -1$ (where "$\equiv_{11}$" denotes congruence modulo 11), it follows that for any nonnegative k,

$$10^k \equiv_{11} (-1)^k.$$

In other words, 10 raised to any even power is 1, and 10 raised to any odd power is -1. Therefore,

$$\begin{aligned} n &= (a_0 10^0 + a_2 10^2 + \cdots a_{d_{even}} 10^{d_{even}}) + (a_1 10^1 + a_3 10^3 + \cdots a_{d_{odd}} 10^{d_{odd}}) \\ &\equiv_{11} (a_0 + a_2 + \cdots a_{d_{even}}) + (-a_1 + -a_3 + \cdots - a_{d_{odd}}) \\ &= s. \end{aligned}$$

Now, since $n \equiv_{11} s$, it follows immediately that n is divisible by 11 if and only if s is.

2.6.1.D

First we apply the Euclidean algorithm to find the greatest common divisor of 34 and 55:

$$\begin{aligned} 55 &= 1 \cdot 34 + 21, \\ 34 &= 1 \cdot 21 + 13, \\ 21 &= 1 \cdot 13 + 8, \\ 13 &= 1 \cdot 8 + 5, \\ 8 &= 1 \cdot 5 + 3, \\ 5 &= 1 \cdot 3 + 2, \\ 3 &= 1 \cdot 2 + 1. \end{aligned}$$

Thus $\gcd(34, 55) = 1$. Now we reverse the steps, in order to express 1 as an integer linear combination of 34 and 55:

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - 1 \cdot (5 - 1 \cdot 3) \\ &= -1 \cdot 5 + 2 \cdot 3 \\ &= -1 \cdot 5 + 2 \cdot (8 - 1 \cdot 5) \\ &= 2 \cdot 8 - 3 \cdot 5 \end{aligned}$$

$$
\begin{aligned}
&= 2 \cdot 8 - 3 \cdot (13 - 1 \cdot 8) \\
&= -3 \cdot 13 + 5 \cdot 8 \\
&= -3 \cdot 13 + 5 \cdot (21 - 1 \cdot 13) \\
&= 5 \cdot 21 - 8 \cdot 13 \\
&= 5 \cdot 21 - 8 \cdot (34 - 1 \cdot 21) \\
&= -8 \cdot 34 + 13 \cdot 21 \\
&= -8 \cdot 34 + 13 \cdot (55 - 1 \cdot 34) \\
&= 13 \cdot 55 - 21 \cdot 34.
\end{aligned}
$$

Thus $1 = 13 \cdot 55 + -21 \cdot 34$.

3.1.21

Assume that $a^2 \equiv_p b^2$. Then $p|(a^2 - b^2) = (a-b)(a+b)$. Because p is prime, we conclude that $p|(a-b)$ (in which case $a \equiv_p b$) or $p|(a+b)$ (in which case $a \equiv_p -b$). Thus, if $a^2 \equiv_p b^2$, then $a \equiv_p \pm b$. For the converse, assume that $a \equiv_p \pm b$. Squaring both sides, we get $a^2 \equiv_p b^2$, as desired.

3.2.23

We know that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. Therefore

$$
\begin{aligned}
\sum_{k=100}^{200} k &= \sum_{k=1}^{200} k - \sum_{k=1}^{99} k \\
&= \frac{1}{2}200 \cdot 201 - \frac{1}{2}99 \cdot 100 \\
&= 20100 - 4950 \\
&= 15150.
\end{aligned}
$$

3.3.36

We wish to prove that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$ for all $n \geq 1$, using mathematical induction. First, when $n = 1$, the statement reduces to $2 = 2$; thus the base step of the induction holds. Now fix $n \geq 1$, and assume that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$; we wish to show that $\sum_{k=1}^{n+1} k2^k = n2^{n+2} + 2$. Well,

$$
\begin{aligned}
\sum_{k=1}^{n+1} k2^k &= \sum_{k=1}^{n} k2^k + (n+1)2^{n+1} \\
&= (n-1)2^{n+1} + 2 + (n+1)2^{n+1} \\
&= 2n \cdot 2^{n+1} + 2
\end{aligned}
$$

$$= \quad n2^{n+2} + 2,$$

where the second equality holds due to the inductive hypothesis. This completes the inductive step of the proof, establishing the desired result.

3.4.18

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

We wish to show, using mathematical induction, that for all $n \geq 1$,

$$A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}.$$

For the base step, when $n = 1$, we note that

$$\begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = A^1,$$

as desired. Now assume that the statement holds for a fixed $n \geq 1$; we wish to prove that it holds for $n + 1$. To see that it does, notice that

$$\begin{aligned} A^{n+1} &= A^n \cdot A \\ &= \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} f_{n+1} + f_n & f_{n+1} \\ f_n + f_{n-1} & f_n \end{bmatrix} \\ &= \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix} \\ &= \begin{bmatrix} f_{(n+1)+1} & f_{n+1} \\ f_{n+1} & f_{(n+1)-1} \end{bmatrix}. \end{aligned}$$

(Here, the second equality follows from the inductive hypothesis.) This completes the proof by induction.

4.4.28

Let $n \geq 2$ be an integer. We wish to show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

A. Let $S = \{a_1, \ldots, a_{2n}\}$ be an arbitrary set of $2n$ elements. The number of ways to choose 2 elements from S is clearly $\binom{2n}{2}$. On the other hand, we can write S as the union $S = S_1 \cup S_2$, where $S_1 = \{a_1, \ldots, a_n\}$ and $S_2 = \{a_{n+1}, \ldots, a_{2n}\}$ have n elements each. Then there are three broad methods for choosing 2 elements from S. First, we may choose 2 elements from S_1; there are $\binom{n}{2}$ ways to do this. Second, we may choose 2 elements from S_2; again there are $\binom{n}{2}$ ways to do this. Third, we may choose 1 element from each S_i; by the multiplication principle, there are $\binom{n}{1}\binom{n}{1}$ ways to do this. Each choice of 2 elements from S is produced by one and only one of these three methods. Thus we see that

$$\binom{2n}{2} = \binom{n}{2} + \binom{n}{2} + \binom{n}{1}\binom{n}{1} = 2\binom{n}{2} + n^2.$$

B. By the definition of $\binom{m}{k}$ and basic algebra,

$$\begin{aligned} \binom{2n}{2} &= \frac{(2n)!}{2!(2n-2)!} \\ &= \frac{(2n)(2n-1)}{2} \\ &= 2n^2 - n \\ &= n^2 - n + n^2 \\ &= 2\frac{n(n-1)}{2} + n^2 \\ &= 2\frac{n!}{2!(n-2)!} + n^2 \\ &= 2\binom{n}{2} + n^2. \end{aligned}$$

4.5.18

We wish to determine how many of strings of 20 decimal digits there are that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s. By Theorem 3 of Section 4.5, the answer is

$$\frac{20!}{2!4!3!1!2!3!2!3!}.$$

(This can be simplified, but it remains a pretty nasty number.)