Geology 376-2, Spring 2004, Assignment 4

You are encouraged to work with others, but you should compose and submit your solutions independently. Give exact answers, and show your work. Let me know if you find any errors.

1. Consider the vector field

$$F(x,y,z) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle.$$

A. Show that curl $F = 0$.

B. Find a potential function for F. You will need to know the following formulae from single-variable calculus: For any constant a,

$$\begin{aligned} \frac{d}{dx}\tan^{-1}(x) &= \frac{1}{1+x^2}, \\ \int \frac{1}{a^2+x^2}\,dx &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C. \end{aligned}$$

2. The flow of a fluid in $\mathbb{R}^3$ is described by a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, which gives the velocity of the flow at each point. Let S be any imaginary surface situated in the flow. (Or, if you prefer, imagine that it is the surface of a net or filter, through which the fluid passes freely.) We declare one side of S to be the "outside", and the other side to be the "inside".

We wish to compute the *flux* of F across S, which measures how much fluid is passing through S, from the inside to the outside. It is defined as a surface integral

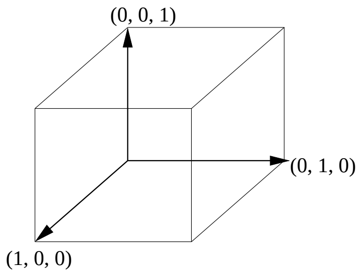
$$\iint_S F \cdot n \, dS.$$

Here, the notation "$\iint_S \ldots dS$" means that we are computing a double integral over the surface S. (More on this later.) Meanwhile, n is the unit outward-pointing normal of S. This means that n is a vector field, defined at every point of S, that's of length 1, perpendicular to S, and pointing outward. Since F and n are vector fields, $F \cdot n$ is a scalar field.

In this exercise, F will be the vector field

$$F(x,y,z) = \left\langle x^2, xy+1, -2z^3 \right\rangle.$$

To keep things simple, S will be the surface of the unit cube drawn below. We'll use the intuitive notion of "outside" and "inside". For example, the unit outward-pointing normal n is $\langle 0,0,1 \rangle$ everywhere on the top face of the cube. On the bottom of the cube, n is $\langle 0,0,-1 \rangle$. On the side of the cube facing us, n is $\langle 1,0,0 \rangle$. You can figure out the other three sides. (On the edges of the cube n is undefined, but that's okay, because the edges "have no area".)



The surface S of the cube is made up of six sides. The total flux across S will be the sum of the fluxes on each side. For an example, let's compute the flux across the top side. Here the normal n is $\langle 0,0,1\rangle$, so $F\cdot n = -2z^3$. Furthermore, at every point on the top of the cube, $z=1$, so $F\cdot n=-2$. Thus the flux across the top of the cube is $\iint_S -2\ dS$. Now we express dS as $dx\ dy$ (or $dy\ dx$), since the top of the cube is parallel to the x-y-plane. Its shadow on the x-y-plane is the rectangle where x runs from 0 to 1 and y runs from 0 to 1. Consequently,

$$\begin{aligned}\iint_S -2\ dS &= \int_0^1\int_0^1 -2\ dx\ dy\\ &= \int_0^1 [-2x]_{x=0}^{x=1}\ dy\\ &= \int_0^1 -2\ dy\\ &= -2.\end{aligned}$$

A similar process can be used for each of the other five sides. For example, on the side facing us, the flux is $\int_0^1\int_0^1 1\ dy\ dz$.

A. Compute the total flux across S by computing the flux on each side and adding them up.

B. Let R be the region of space inside the cube. Then the divergence theorem says that

$$\iint_S F\cdot n\ dS = \iiint_R \text{div } F\ dV.$$

The "dV" means "with respect to volume". Turn the dV into something like $dx\ dy\ dz$, set up bounds for x, y, and z, calculate the divergence of F, and compute the triple integral. You should get the same answer as in part A.

C. Is this flow incompressible? Is more fluid flowing into the cube or out of it?