Math 32-05/06, Fall 2005, Exam 4 Answers

1. In general, what does $\sum_{n=0}^{\infty} a_n = A$ mean? That is, how is the value of a series defined?

(This is repeated verbatim from the third exam.) The value of a series is the limit of the sequence of its partial sums, if it exists. That is, $\sum_{n=0}^{\infty} a_n = A$ means that $\lim_{k\to\infty} \sum_{n=0}^{k} a_n = A$.

2. Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{n^3}.$$

(This is 11.8 #7.) We proceed by the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}x^{n+1}n^3}{(n+1)^3 3^n x^n} = \frac{3n^3}{n^3+3n^2+3n+1}|x|.$$

As $n \to \infty$, this goes to $3|x|$, which we want to be less than 1. Therefore the series converges for x in $(-1/3, 1/3)$. At $x = 1/3$ the series is $\sum 1/n^3$, which converges since it's a p-series with $p = 3 > 1$. At $x = -1/3$ the series is $\sum(-1)^n/n^3$, which converges by the alternating series test. So the interval of convergence is $[-1/3, 1/3]$.

3. A. Write $\ln 3/2$ as a series.

(Most of this whole problem was done in class during our review.) Since $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$,

$$\ln 3/2 = \ln(1+1/2) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}.$$

B. How many terms are required to approximate $\ln 3/2$ with an error smaller than 10^{-2}?

The error in truncating after term n (which we will call a_n) has size $|R_n| < |a_{n+1}| = \frac{1}{(n+1)2^{n+1}}$. So choosing $n = 3$ produces error less than $1/64$, whereas $n = 4$ produces error less than $1/160$. Therefore the approximation to within 10^{-2} must use all four terms up to and including a_4.

C. What is the value of the approximation with that many terms?

The approximation is 1/2 - 1/8 + 1/24 - 1/64 = 77/192.

D. What happens if you use the same method to compute $\ln 5/2$? Explain in detail.

The series is

$$\ln 5/2 = \ln(1+3/2) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n}{n2^n},$$

which diverges by the nth term test (it can easily be shown). This is not surprising, since $x = 3/2$ lies outside the interval of convergence of $\ln(1+x)$.

4. For each series, determine whether it diverges, converges conditionally, or converges absolutely. Explain your answers.

A.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^n}{3^{n^2}}$$

(This is 11.7 #42.) We proceed by the root test.

$$|a_n|^{1/n} = \left(\frac{n^n}{3^{n^2}}\right)^{1/n} = \frac{n}{3^n}.$$

So

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{x\to\infty} \frac{x}{3^x} = \lim_{x\to\infty} \frac{1}{3^x \ln 3} = 0.$$

(Here we have used L'Hôpital's rule.) So the root test implies that the series converges absolutely.

B.

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$$

Notice that $\cos(\pi n) = (-1)^n$, so the series is alternating. The absolute value of the nth term is $1/\sqrt{n}$, which decreases to 0, so the series converges by the alternating series test. However, the series $\sum 1/\sqrt{n}$ diverges, because it is the p-series with $p = 1/2 \leq 1$. Therefore the series as written converges conditionally.

5. Solve the differential equation $y' - xy = 0$ using the method of power series. If your series answer is a function that you recognize, then express the function in closed form as well.

Say $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

and

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n.$$

So

$$y' - xy = a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = a_1 + \sum_{n=1}^{\infty} \left((n+1) a_{n+1} - a_{n-1}\right) x^n.$$

Setting this equal to $0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$ and comparing coefficients, we find that $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n \geq 1$. The first coefficient, a_0, is undetermined. The odd-numbered coefficients $a_1, a_3, a_5, \ldots$ are all 0, and the even-numbered coefficients are

$$a_2 = a_0/2, a_4 = a_2/4 = a_0/(4 \cdot 2), a_6 = a_4/6 = a_0/(6 \cdot 4 \cdot 2), \ldots,$$

which seem to follow the pattern

$$a_{2k} = \frac{a_0}{(2k)(2k-2)\cdots(4)(2)} = \frac{a_0}{k! 2^k}.$$

This pattern holds even for the $n = 2k = 0$ term. Therefore

$$y = \sum_{k=0}^{\infty} a_{2k} x^{2k} = \sum_{k=0}^{\infty} \frac{a_0}{k! 2^k} x^{2k}.$$

We recognize this as

$$y = a_0 \sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} = a_0 e^{x^2/2}.$$

6. Find the Taylor series for $f(x) = \ln(1+x)$ centered at $a = 0$ explicitly, by taking successive derivatives and using Taylor's formula.

Differentiating, we find $f^{(1)}(x) = (1+x)^{-1}$, $f^{(2)}(x) = -(1+x)^{-2}$, $f^{(3)}(x) = 2(1+x)^{-3}$, $f^{(4)}(x) = -6(1+x)^{-4}$, $f^{(5)}(x) = 24(1+x)^{-5}$, and so on. So it appears that, for $n \geq 1$, $f^{(n)}(x) = (-1)^{n+1}(n-1)!(1+x)^{-n}$ and $f^{(n)}(0) = (-1)^{n+1}(n-1)!$. When $n = 0$, we have $f^{(0)}(0) = f(0) = \ln 1 = 0$, so the degree-0 term in the Taylor series is 0. Therefore the Taylor series is

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!x^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

7. Use power series to compute $\int_0^1 \sin(x^2)\,dx$. You should leave your answer as a series, simplified as much as possible.

We know that $\sin x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$, so $\sin x^2 = \sum_{n=0}^{\infty}(-1)^n x^{4n+2}/(2n+1)!$, so

$$\int \sin x^2\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!},$$

so

$$\int_0^1 \sin x^2\,dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}.$$