Using Taylor Polynomials to Approximate Functions

1. (a) Find $P_3(x)$, the third-order Taylor polynomial for $f(x) = e^x$ at $a = 0$.
(Ans: $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.)

(b) Find an upper bound for the error if $P_3(.5)$ is used to approximate $e^{.5}$, given that $e^{.5} < 4^{.5} = 2$.
(Ans: Error $= |R_3(.5)| < \frac{1}{4!\,2^3} < .00521$.)

(c) Find n so that $P_n(.5)$, the n^{th}-order Taylor polynomial for $f(x) = e^x$ at $a = 0$ with $x = .5$, estimates $e^{.5}$ to five decimal places (i.e., with error $< .000005$). For the n you find, evaluate $P_n(.5)$.
(Ans: $n = 6$ and $e^{.5} \approx P_6(.5) \approx 1.64872$ to five-decimal-place accuracy.)

2. (a) Find $P_4(x)$, the fourth-order Taylor polynomial for $f(x) = \ln x$ at $a = 1$.
(Ans: $P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$.)

(b) Find an upper bound for the error if $P_4(x)$ is used to approximate $\ln x$ for $|x - 1| < .1$.
(Ans: Error $= |R_4(x)| < \frac{1}{9^5 5} < 3.39 \times 10^{-6}$.)

3. Find an upper bound for the error if $x - \frac{x^3}{3!}$ is used to approximate $\sin(x)$ for $x = 31^\circ = \frac{31\pi}{180}$.
(Ans: If we regard the polynomial as $P_3(x)$, then the

$$\text{error} = |R_3(\tfrac{31\pi}{180})| \le \frac{\left(\frac{31\pi}{180}\right)^4}{4!} < .00357.$$

If we view $x - \frac{x^3}{3!}$ as $P_4(x)$, then the error $= |R_4(\frac{31\pi}{180})| \le \frac{\left(\frac{31\pi}{180}\right)^5}{5!} < .000387$.
We can also get the latter upper bound by using the Alternating Series Error Estimate.)

4. Find an upper bound for the error if $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ is used to approximate $\sin(x)$ for $x = 31^\circ = \frac{31\pi}{180}$.
(Ans: Regarding the polynomial as $P_8(x)$ we have the

$$\text{error} = |R_8(\tfrac{31\pi}{180})| \le \frac{\left(\frac{31\pi}{180}\right)^9}{9!} < 1.10 \times 10^{-8}.$$

This upper bound can also be obtained with the Alternating Series Error Estimate.)

5. Let $P_n(x)$ be the n^{th}-order Taylor polynomial for $f(x) = sin\, x$ at $a = \frac{\pi}{6}$.

(a) Find $P_3(x)$.
(Ans: $P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{1}{2}\frac{(x-\frac{\pi}{6})^2}{2!} - \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{6})^3}{3!}$.)

(b) Find an upper bound for the error when $P_3(\frac{31\pi}{180})$ is used to approximate $sin(31^\circ) = sin(\frac{31\pi}{180})$
(Ans: $|R_3(\frac{31\pi}{180})| \leq \frac{(\frac{\pi}{180})^4}{4!} < 3.87 \times 10^{-9}$.)

(c) Find an upper bound for the error when $P_7(\frac{31\pi}{180})$ is used to approximate $sin(31^\circ) = sin(\frac{31\pi}{180})$.
(Ans: Error = $|R_7(\frac{31\pi}{180})| \leq \frac{(\frac{\pi}{180})^8}{8!} < 2.14 \times 10^{-19}$.)

Note: Problems 3–5 illustrate that approximations improve when n increases and when a is chosen closer to x.