Name:

I have adhered to the Duke Community Standard.

Signature:

Math 31L 03-04 Fall 2006 Exam 1

Instructions: You have 70 minutes. You may use your TI-83 or equivalent calculator. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified answers, as exact as possible. Put a box around each answer. Ask questions if any problem is unclear. Good luck.

1. In each of the following, draw a graph $y = f(x)$ with the specified properties, or explain briefly why no such graph can exist.

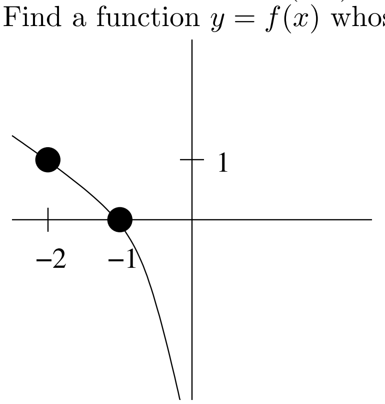
A. $f(x)$, $f'(x)$, and $f''(x)$ are all negative for all x.

Solution: Try $f(x) = -e^x$.

B. f is differentiable, $f(x)$ is always positive, and $f''(x)$ is negative for $x < 0$.

Solution: Try $f(x) = \frac{(x-1)^2}{(x-1)^2+1} + 1$.

2. Find a function $y = f(x)$ whose graph could be the one depicted below.



Solution: $y = \log_2(-x)$.

3. Compute the derivatives of the following functions.

A. $f(x) = 3x^5 - 2x$

Solution: $f'(x) = 15x^4 - 2$.

B. $f(x) = x^e - e^x$

Solution: $f'(x) = ex^{e-1} - e^x$.

C. $f(x) = 3^x \ln 3$

Solution: $f'(x) = 3^x(\ln 3)^2$.

D. $f(x) = \frac{x^4+1}{ex^4-2x}$

Solution: $f'(x) = \frac{-6x^4-4ex^3+2}{(ex^4-2x)^2}$. [Certainly show work on this one!]

E. $f(x) = 2^7$

Solution: $f'(x) = 0$.

4. Climate scientists in the German Alps have been studying Musterhorn Glacier B, which I just made up. This is a giant, rectangular block of ice. Let w denote its width, h its height, and ℓ its length, all in kilometers. Its volume is $v = wh\ell$. Its height and length have been decreasing over recent years due to climate change, as the chart below shows. Its width is a constant 1.31 km, because it is trapped in a steep valley between two mountains.

Year t	Height $h(t)$ (in km)	Length $\ell(t)$ (in km)
1980	0.82	2.15
1990	0.80	2.12
2000	0.79	2.10

A. Numerically estimate $h'(t)$ at $t = 1990$.

Solution: $h'(1990) \approx (0.79 - 0.80)/10 = -0.01/10 = -0.001$ km/yr. [You could also estimate based on the 1980 and 1990 figures, or average the two approaches.]

B. What does the quantity that you computed in Part A mean about the glacier? What are its units?

Solution: It is the rate at which the height is increasing, in km/yr. It's negative, since the height is decreasing.

C. Numerically estimate $\ell'(t)$ at $t = 1990$.

Solution: $\ell'(1990) \approx (2.10 - 2.12)/10 = -0.02/10 = -0.002$ km/yr.

D. Using your answers to Parts A and C, and the product rule, estimate the rate of change of the volume of the glacier in 1990.

Solution: $v(t) = wh(t)\ell(t)$, so $v'(t) = wh'(t)\ell(t)+wh(t)\ell'(t)$. So $v'(1990) \approx (1.31)(-0.001)(2.12) + (1.31)(0.80)(-0.002) = -0.0048732$ km^3/yr.

E. What was the volume v of the glacier in 1990? In 2000? Using these, make another estimate the rate of change of the volume in 1990.

Solution: $v(1990) = (1.31)(0.80)(2.12) = 2.22176$, $v(2000) = (1.31)(0.79)(2.10) = 2.17329$, and so $v'(1990) \approx (2.17329 - 2.22176)/10 = -0.004847$ km^3/yr.

5. This problem deals with an unknown function $y = f(x)$. All we know is that $f'(x) = \frac{1}{2-x^2}$ and that $f(0) = 3$.

A. What is the linear approximation to $f(x)$ at $x = 0$?

Solution: The approximating line has slope $f'(0) = 1/2$ and passes through $(0, 3)$, so it is given by $y = x/2 + 3$.

B. Using the linear approximation of Part A, estimate $f(1)$.

Solution: At $x = 1$ the approximation has value $1/2 + 3 = 3.5$.

C. Using your answer to Part B, find a linear approximation to $f(x)$ at $x = 1$.

Solution: The approximating line has slope $f'(1) = 1$ and passes through $(1, 3.5)$, so it is given by $y = x + 2.5$.

D. Using your answer to Part C, estimate $f(2)$.

Solution: At $x = 2$ the approximation has value $2 + 2.5 = 4.5$.

E. Using your results thus far, sketch an approximate graph of $y = f(x)$ for $0 \leq x \leq 2$.

Solution: [Plot points $(0, 3)$, $(1, 3.5)$, and $(2, 4.5)$, and connect them with lines.]

F. The approximation goes bad somewhere between $x = 1$ and $x = 2$. Why?

Solution: The derivative f' has a vertical asymptote at $x = \sqrt{2} \approx 1.4$. As x approaches this asymptote from the left, the slope of $y = f(x)$ shoots to infinity, causing that graph to diverge wildly from our linear approximation. (It also behaves badly from the right.) Our step size of $\Delta x = 1$ is too large to detect this behavior; moving to a smaller step size would help, but the fact that f is not differentiable means that Euler's method will never describe it really precisely.

6. Use the definition of the derivative to compute the derivative of the function $f(x) = \frac{1}{2x-3}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1/(2x+2h-3) - 1/(2x-3)}{h} \\ &= \lim_{h\to 0} \frac{2x-3-2x-2h+3}{h(2x+2h-3)(2x-3)} \\ &= \lim_{h\to 0} \frac{-2}{(2x+2h-3)(2x-3)} \\ &= \frac{-2}{(2x-3)^2} \end{aligned}$$

7. Find the following limits or explain briefly why they do not exist.

A. $\lim_{x\to 0} \frac{3x^2-4x+1}{-2x^2+2}$

Solution: [Plug in $x = 0$ to get a limit of 1/2.]

B. $\lim_{x\to \infty} \frac{3x^2-4x+1}{-2x^2+2}$

Solution: [Divide top and bottom by x^2. Then most terms go to 0; what remains is $-3/2$.]

C. $\lim_{x\to 1} \frac{3x^2-4x+1}{-2x^2+2}$

Solution: [Divide top and bottom by $x - 1$; what remains goes to $-1/2$ by plugging in $x = 1$.]