Name:

I have adhered to the Duke Community Standard.

Signature:

Math 31L 03-04 Fall 2006 Exam 2

Instructions: You have 60 minutes. You may use your TI-83 or equivalent calculator. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified answers, as exact as possible. Put a box around each answer. Ask questions if any problem is unclear. Good luck.

1. Compute

$$\lim_{x\to 0} \frac{x}{\arctan x}.$$

2. We have studied the differential equations

$$dA/dt = -k_1 A + k_2 B, \quad (1)$$
$$dB/dt = k_1 A - k_2 B, \quad (2)$$

where k_1 and k_2 are positive constants. In this problem, assume that dA/dt is always positive, as well. To answer the following questions, you do *not* need to solve the differential equations.

A. Show that dB/dt is always negative.

B. Differentiate Equation (1) and show that the second derivative $\frac{d^2A}{dt^2}$ is always negative.

C. At what time $t \geq 0$ is A increasing fastest?

D. Sketch the graph of dA/dt, as a function of t.

3. Suppose that the engine in an automobile burns 1 liter of fuel per hour just to keep itself running (even when the automobile is not moving). When the automobile is going v km/h, the engine burns an *additional* $0.001v^2$ liters per hour to maintain that speed.

I need to drive 750 km, and I'd like to use as little fuel as possible. By the way, I never exceed the speed limit, which is 90 km/h.

A. If I drive at speed v, how many liters of fuel do I burn on my trip? Give your answer as a function $f(v)$. (Hint: How many hours long is the drive?)

B. We are going to minimize $f(v)$ on what closed, bounded interval?

C. Find the global minimum.

4. An object of mass m is moving around; its position at time t is given by

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{k}{m}}\,t\right),$$

where A, B, and k are positive constants.

A. Show that x satisfies the differential equation $x'' = -\frac{k}{m}x$.

B. Using the differential equation from Part A, find a simple expression for the force F that must be acting on the object, in terms of x.

C. In this part only, assume $B = 0$. At what time t is the force greatest?

5. A water tank is in the shape of a cone, with its tip pointing down. Its height is 10 meters and its top radius is 8 meters. Water is flowing into the tank at 0.1 cubic meters per minute and also leaking out at a rate of $0.001h^2$ cubic meters per minute, where h is the depth of the water of the tank in meters. Let v be the volume of the water in the tank.

A. What is dv/dt, in terms of h?

B. What is dv/dh, in terms of h?

C. Does the tank ever overflow? Explain.

6. Recall that $\cot x = \cos x / \sin x$. Even if you have the answers to this problem memorized, you must still show how they are derived.

A. Find the derivative of $\cot x$.

B. Find the derivative of $\operatorname{arccot} x$. Simplify as much as possible.