Name:

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Signature:

Math 31L 03-04 Fall 2006 Exam 2

Instructions: You have 60 minutes. You may use your TI-83 or equivalent calculator. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified answers, as exact as possible. Put a box around each answer. Ask questions if any problem is unclear. Good luck.

1. Compute

$$\lim_{x\to 0} \frac{x}{\arctan x}.$$

Solution: [Use L'Hopital's rule to get to

$$\lim_{x\to 0} \frac{1}{\frac{1}{1+x^2}} = \lim_{x\to 0} 1 + x^2;$$

then plug in $x = 0$ to get 1.]

2. We have studied the differential equations

$$dA/dt = -k_1 A + k_2 B, \quad (1)$$
$$dB/dt = k_1 A - k_2 B, \quad (2)$$

where k_1 and k_2 are positive constants. In this problem, assume that dA/dt is always positive, as well. To answer the following questions, you do *not* need to solve the differential equations.

A. Show that dB/dt is always negative.

Solution: $dB/dt = k_1 A - k_2 B = -(-k_1 A + k_2 B) = -dA/dt$; since dA/dt is positive, dB/dt must be negative.

B. Differentiate Equation (1) and show that the second derivative $\frac{d^2A}{dt^2}$ is always negative.

Solution: $\frac{d^2A}{dt^2} = -k_1 dA/dt + k_2 dB/dt$; since k_1 and dA/dt are positive, the first term is negative; since k_2 is positive and dB/dt negative, the second term is negative. Thus $\frac{d^2A}{dt^2}$ is negative.

C. At what time $t \geq 0$ is A increasing fastest?

Solution: We wish to find where dA/dt is maximized. By Part B, dA/dt is always decreasing, so it is maximized at the earliest time $t \geq 0$, namely $t = 0$.

D. Sketch the graph of dA/dt, as a function of t.

Solution: [Your graph should begin somewhere on the positive (dA/dt)-axis, and then gently slope downward, always remaining above the t-axis, approaching horizontal.]

3. Suppose that the engine in an automobile burns 1 liter of fuel per hour just to keep itself running (even when the automobile is not moving). When the automobile is going v km/h, the engine burns an *additional* $0.001v^2$ liters per hour to maintain that speed.

I need to drive 750 km, and I'd like to use as little fuel as possible. By the way, I never exceed the speed limit, which is 90 km/h.

A. If I drive at speed v, how many liters of fuel do I burn on my trip? Give your answer as a function $f(v)$. (Hint: How many hours long is the drive?)

Solution: I drive $750/v$ hours, and I use $1 + 0.001v^2$ liters per hour, so I use

$$f(v) = (750/v)(1 + 0.001v^2) = 750v^{-1} + 0.75v$$

liters on my trip.

B. We are going to minimize $f(v)$ on what closed, bounded interval?

Solution: $[0, 90]$. [The function is not continuous at $v = 0$, but I can still talk about a minimization problem on that interval. In particular, no other closed, bounded interval makes sense. For example, $[1, 90]$ doesn't make sense, because why shouldn't I be allowed to drive 1/2 km/h?]

C. Find the global minimum.

Solution: The derivative is $f'(v) = -750v^{-2} + 0.75$. This is zero at $v = 10\sqrt{10} \approx 32$, with value $f(10\sqrt{10}) = 15\sqrt{10} \approx 47$. It is undefined at $v = 0$, which is also an endpoint. As $v \to 0^+$, $f(v)$ clearly increases to infinity, so $v = 0$ cannot be the min. [Driving at speed 0 means that you take infinite time, hence use infinite fuel, just keeping the engine running.] The other endpoint, $v = 90$, produces $f(v) \approx 76$. So the minimum must occur at $v = 10\sqrt{10} \approx 32$ with value $f(10\sqrt{10}) = 15\sqrt{10} \approx 47$.

4. An object of mass m is moving around; its position at time t is given by

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\, t\right) + B\sin\left(\sqrt{\frac{k}{m}}\, t\right),$$

where A, B, and k are positive constants.

A. Show that x satisfies the differential equation $x'' = -\frac{k}{m}x$.

Solution: [Differentiate $x(t)$ twice. Also, multiply $x(t)$ by $-k/m$. See that the two are equal.]

B. Using the differential equation from Part A, find a simple expression for the force F that must be acting on the object, in terms of x.

Solution: We know $F = ma$ and $a = x'' = -k/mx$. So $F = -kx$.

C. In this part only, assume $B = 0$. At what time t is the force greatest?

Solution: $F = -kA\cos(\sqrt{k/m}t)$ is a negative multiple of $\cos(\sqrt{k/m}t)$, so it is maximized when $\cos(\sqrt{k/m}t)$ is minimized, which is when $\sqrt{k/m}t = \pi + n2\pi$. That is,

$$t = \sqrt{m/k}\pi + n2\sqrt{m/k}\pi.$$

[The smallest positive solution is $t = \sqrt{m/k}\pi$.]

5. A water tank is in the shape of a cone, with its tip pointing down. Its height is 10 meters and its top radius is 8 meters. Water is flowing into the tank at 0.1 cubic meters per minute and also leaking out at a rate of $0.001h^2$ cubic meters per minute, where h is the depth of the water of the tank in meters. Let v be the volume of the water in the tank.

A. What is dv/dt, in terms of h?

Solution: [By the way, this problem is essentially 4.6#32.] $dv/dt = 0.1 - 0.001h^2$.

B. What is dv/dh, in terms of h?

Solution: $v = \pi r^2 h/3$. By similar triangles we know $r = 4h/5$, so $v = (16\pi/75)h^3$, so

$$dv/dh = (16\pi/25)h^2 = 0.64\pi h^2.$$

C. Does the tank ever overflow? Explain.

Solution: [There are several ways to do this.] By the chain rule, $dv/dt = dv/dh \cdot dh/dt$. Plugging in the answers to Parts A and B, we can solve for

$$\frac{dh}{dt} = \frac{0.1 - 0.001h^2}{0.64\pi h^2} = \frac{0.1}{0.64\pi}(\frac{1}{h^2} - 0.01).$$

This is positive for $h < 10$, zero at $h = 10$, and negative for $h > 10$. This means that the water fills up to $h = 10$ but does not ever go over that. So it does not overflow.

6. Recall that $\cot x = \cos x/\sin x$. Even if you have the answers to this problem memorized, you must still show how they are derived.

A. Find the derivative of $\cot x$.

Solution:

$$\frac{d}{dx}\frac{\cos x}{\sin x} = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x}.$$

B. Find the derivative of $\operatorname{arccot} x$. Simplify as much as possible.

Solution: $\cot(\text{arccot } x) = x$, so by differentiation (using the result of Part A) we have

$$\frac{-1}{\sin^2(\text{arccot } x)} \cdot \frac{d}{dx}\text{arccot } x = 1.$$

Now one can show using the Pythagorean theorem that

$$\sin(\text{arccot } x) = \frac{1}{\sqrt{1+x^2}}.$$

So

$$\frac{d}{dx}\text{arccot } x = -\sin^2(\text{arccot } x) = \frac{-1}{1+x^2}.$$