Curves.

Let n be a integer which is greater than or equal 2.

We say $\mathbf{P}$ is a **curve in $\mathbf{R}^n$** if, for some open interval I in $\mathbf{R}$,

$$\mathbf{P} : I \to \mathbf{R}^n$$

and $\mathbf{P}$ is continuous. Think of $\mathbf{P}$ as standing for **position**.

In order to work with curves it necessary to assume some differentiability as well. When they exist, We call $\mathbf{P}'$ and $\mathbf{P}''$ the **velocity** and **acceleration** of $\mathbf{P}$, respectively. We say the curve $\mathbf{Q}$ is a **reparameterization of $\mathbf{P}$** if there exists a continuous coordinate u on I such that

$$\mathbf{Q} \circ u \; = \; \mathbf{P};$$

note that the range of $\mathbf{P}$ is the same as the range of $\mathbf{Q}$. In some contexts it is only the range of $\mathbf{P}$ that is important. Recall, for example, in the supplement about coordinates we set

$$S \; = \; \{(a,b) \in \mathbf{R}^2 : a^2 + b^2 = 1, \; a > 0, \; b > 0\}.$$

It would be reasonable to call S a curve, but it is not a curve in the sense of the above definition. If x, y, θ and u are as in that supplement then $x^{-1}, y^{-1}, \theta^{-1}$ and u^{-1} are all curves with range S and with domain $(0,1), (0,1), (0, \frac{\pi}{2}$ and $(0, \infty)$, and they are all reparamterizations of each other.

Arclength and the unit tangent vector. Let us now assume that the velocity of P exists and never vanishes. Let t be the function on I such that $t(\tau) = \tau$ whenever $\tau \in I$. Note that t is a coordinate on I. Suppose $s : I \to \mathbf{R}$ is such that

$$\frac{ds}{dt} \; = \; |\frac{d\mathbf{P}}{dt}|;$$

Because $\frac{ds}{dt} > 0$ it follows that that s is a coordinate on I which we call an **arclength coordinate for $\mathbf{P}$**. Because any two arclength coordinates differ by a constant, it follows from the chain rule that

$$\frac{d}{ds_1} \; = \; \frac{d}{ds_2}$$

whenever s_1 and s_2 are arclength coordinates for $\mathbf{P}$. Any arclength coordinate s is obtained by picking a point t_0 in I and a real number s_0 and setting

$$s(t) \; = \; s_0 + \int_{t_0}^{t} |\frac{d\mathbf{P}}{dt}|(\tau)\, d\tau$$

for t in I.

Fix an arclength coordinate s for $\mathbf{P}$. We define the **unit tangent vector $\mathbf{T}$ of $\mathbf{P}$** by setting

$$\mathbf{T} \; = \; \frac{d\mathbf{P}}{ds} \; = \; \frac{dt}{ds}\frac{d\mathbf{P}}{dt} \; = \; |\mathbf{P}|^{-1}\frac{d\mathbf{P}}{dt}.$$

Note that $\mathbf{T}$ is independent of the choice of arclength coordinate and that, by the chain rule,

$$|\mathbf{T}| \; = \; |\frac{dt}{ds}\frac{d\mathbf{P}}{dt}| \; = \; 1$$

which explains why "unit" appears above.

The curvature vector. **Next we assume that the acceleration of P exists** and we define the **curvature vector K of P** by setting

$$\mathbf{K} = \frac{d\mathbf{T}}{ds}.$$

Picture this! Inasmuch as $\mathbf{K}$ is a derivative of a unit vector it must be perpendicular to that unit vector;[1] that is,

$$\mathbf{K} \bullet \mathbf{T} = 0.$$

The curvature is **independent of parameterization** in the following sense. Suppose u is twice differentiable coordinate on I, $\frac{du}{dt}$ never vanishes and $\mathbf{Q} = \mathbf{P} \circ u$. Let $\mathbf{L}$ be the curvature vector of $\mathbf{Q}$. Then

$$\mathbf{K} = \mathbf{L} \circ u.$$

We let

$$\kappa = |\mathbf{K}|$$

and call this real variable on I the **curvature of P.**

Suppose $\mathbf{K}$ vanishes identically. Then $\mathbf{T} = \frac{d\mathbf{P}}{ds}$ is constant which implies that

$$\mathbf{P} = \mathbf{P}(t_0) + (s - s(t_0))\mathbf{T}(t_0)$$

whenever $t_0 \in I$; in particular, the range of $\mathbf{P}$ is contained in a straight line. Conversely, if the range of $\mathbf{P}$ is contained in a straight line then it is easy to verify that the curvature is identically zero.

Suppose $\mathbf{K}$ never vanishes. We then define the **unit normal vector N of P** by setting

$$\mathbf{N} = \frac{1}{\kappa}\mathbf{K};$$

evidently, $\mathbf{N}$ is a unit vector.

We have

$$\begin{aligned}
\mathbf{K} &= \frac{d\mathbf{T}}{ds} \\
&= \Big|\frac{d\mathbf{P}}{dt}\Big|^{-1} \frac{d}{dt}\Big(\Big|\frac{d\mathbf{P}}{dt}\Big|^{-1}\frac{d\mathbf{P}}{dt}\Big) \\
&= \Big|\frac{d\mathbf{P}}{dt}\Big|^{-1}\Big(-\Big|\frac{d\mathbf{P}}{dt}\Big|^{-3}\frac{d^2\mathbf{P}}{dt^2} \bullet \frac{d\mathbf{P}}{dt} + \Big|\frac{d\mathbf{P}}{dt}\Big|^{-1}\frac{d^2\mathbf{P}}{dt^2}\Big) \\
&= \Big|\frac{d\mathbf{P}}{dt}\Big|^{-2}\Big(\frac{d^2\mathbf{P}}{dt^2} - \Big(\frac{d^2\mathbf{P}}{dt^2} \bullet \mathbf{T}\Big)\mathbf{T}\Big).
\end{aligned}$$

Rearranging, we obtain the very useful formula

$$\frac{d^2\mathbf{P}}{dt^2} = \Big(\frac{d^2\mathbf{P}}{dt^2} \bullet \mathbf{T}\Big)\mathbf{T} + \Big|\frac{d\mathbf{P}}{dt}\Big|^2 \kappa \mathbf{N}.$$

The case $n = 2$. Suppose $n = 2$ and $\kappa > 0$. Inasmuch as $\mathbf{T} \bullet \mathbf{N} = 0$ we find that

$$\frac{d\mathbf{T}}{ds} \bullet \mathbf{N} + \mathbf{T} \bullet \frac{d\mathbf{N}}{ds} = 0$$

[1] This is the **first basic principle of differential geometry**, and it follows from the extremely useful formula

$$\frac{d}{dt}|\mathbf{X}|^2 = 2\frac{d\mathbf{X}}{dt} \bullet \mathbf{X}.$$

The second and last basic principle of differential geometry is that mixed partial derivatives are equal.

which forces

$$\frac{d\mathbf{N}}{ds} == \kappa\mathbf{T}.$$

If κ is constant we this implies that

$$\frac{d}{ds}(\mathbf{P} + \frac{1}{\kappa}\mathbf{N}) = \mathbf{T} + \frac{1}{\kappa}(-\kappa\mathbf{T}) = \mathbf{0};$$

that is, $\mathbf{P} + \frac{1}{\kappa}\mathbf{N}$ is a constant vector so the range of $\mathbf{P}$ is contained in a circle of radius $\frac{1}{\kappa}$.

The case $n = 3$. Now suppose that $n = 3$ and $\kappa > 0$. We then define the **binormal vector B of P** by setting

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

and the **torsion** τ by setting

$$\tau = \frac{d\mathbf{N}}{ds} \bullet \mathbf{B}.$$

These quantities are easily seen to be indendent of reparameterization by an *increasing* coordinate; if the new coordinate is decreasing, $\mathbf{B}$ and τ change sign. It follows from examination of various dot products that

$$\frac{d}{ds}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}.$$

In particular, if the torsion vanishes identically the binormal vector must be constant so the ranges $\mathbf{T}$ and $\mathbf{N}$ are contained in a plane P passing through $\mathbf{0}$. It follows easily that if and only if the range of $\mathbf{P}$ is a subset of a plane parallel to P.

It is not too hard to show that κ and τ are constant if and only if the range of $\mathbf{P}$ is contained in a helix. In fact, if we set

$$\mathbf{u} = \frac{1}{\sqrt{\kappa^2 + \tau^2}}\big(\tau\mathbf{T} + \kappa\mathbf{B}\big)$$

we find that $\mathbf{u}$ is a constant unit vector and that

$$\frac{d^2}{dt^2}\mathbf{P} \bullet \mathbf{u} = 0.$$

Moreover, the curve $I \ni t \mapsto \mathbf{P}(t) - (\mathbf{P}(t) \bullet \mathbf{u})\mathbf{u}$ is a plane curve whose curvature is easily seen to be constant.

Finally, "curves in $\mathbf{R}^3$ are determined up to rigid motions and reparameterization by their curvature and torsion." This may be a bit beyond the scope of this course because it depends on the uniquness theorem for ordinary differential equations so we will not go into it further. What's more, I'm getting tired of writing all this stuff that ought to be laid out in the book anyway.