Math 103-03, Spring 2006, Exam 1

Name:

I have adhered to the Duke Community Standard in completing this examination.

Signature:

Instructions: You have 50 minutes. Calculators are not allowed. Always **show all of your work**. Pictures are often helpful. Partial credit may be awarded. Give **simplified**, **exact** answers, and make sure they are clearly marked.

1 (12 pts). Let $\vec{u} = \langle 3, \pi, -1\rangle, \vec{v} = \langle 2, 1, -3\rangle$ be vectors in $\mathbf{R}^3$. Compute the following quantities. Write your final answers in the spaces provided. (Here $\text{comp}_{\vec{v}}\vec{u}$ is as defined in the book; it is the length of the vector $\vec{u}^{||} = \text{proj}_{\vec{v}}\vec{u}$ discussed in class.)

Solution:

$\vec{u} + \vec{v} = \langle 5, \pi + 1, -4\rangle$

$2\vec{u} = \langle 6, 2\pi, -2\rangle$

$6\vec{u} + -4\vec{v} = \langle 10, 6\pi - 4, 6\rangle$

$\vec{u} \cdot 2\vec{v} = 18 + 2\pi$

$\text{comp}_{\vec{v}}\vec{u} = (9 + \pi)/\sqrt{14}$

$\vec{u} \times \vec{v} = \langle -3\pi + 1, 7, 3 - 2\pi\rangle$

2 (8 pts). Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be any three nonzero vectors in $\mathbf{R}^3$. Under what conditions will it be true that $|\vec{u} \times (\vec{v} \times \vec{w})| = |\vec{u}||\vec{v}||\vec{w}|$? Explain in detail, including the geometric meaning of your answer.

Solution: By the geometric version of the cross product, $|\vec{u} \times (\vec{v} \times \vec{w})| = |\vec{u}||\vec{v} \times \vec{w}| \sin\theta = |\vec{u}||\vec{v}||\vec{w}| \sin\phi \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v} \times \vec{w}$ and ϕ is the angle between $\vec{v}$ and $\vec{w}$. So we require $\sin\theta \sin\phi = 1$, so $\sin\theta = \sin\phi = 1$ (since we deal only with angles between 0 and π). This means that $\vec{u}$ is perpendicular to $\vec{v} \times \vec{w}$ and $\vec{v}$ is perpendicular to $\vec{w}$. One could also say that $\vec{v}$ and $\vec{w}$ are perpendicular and $\vec{u}$ is in the plane through $\vec{v}$, $\vec{w}$, and $\vec{0}$.

3 (8 pts). Let $\vec{a}$ and $\vec{b}$ be two distinct points in $\mathbf{R}^3$. (Here, as usual, we are identifying a vector with the point it points to when its tail is at the origin.) The *perpendicular bisector* of the line segment from $\vec{a}$ to $\vec{b}$ is the plane through the midpoint of the line segment that is perpendicular to the line segment. Find an equation for the perpendicular bisector.

Solution: The midpoint is $\vec{m} = (\vec{a} + \vec{b})/2$. The vector $\vec{b} - \vec{a}$ from $\vec{a}$ to $\vec{b}$ is normal to the plane. So an equation for the plane is $(\vec{b} - \vec{a}) \cdot (\vec{x} - (\vec{a} + \vec{b})/2) = 0$.

4 (12 pts). An airplane on a mission to drop humanitarian aid packages over a war-torn region is flying directly east at a speed of 40 m/s and at an altitude of 1000 meters. Suddenly the copilot spots a needy person named Trevor directly below the plane and releases a package for him. The package accelerates down at 10 m/s^2 due to gravity. A wind blowing southeast also accelerates the package at 2 m/s^2. How far from Trevor does the package land? Write your answer in the space provided.

Solution: Putting Trevor at the origin and the positive x- and y-axes pointing east and north as usual, we have initial conditions $\vec{x_0} = \langle 0, 0, 1000\rangle$ and $\vec{v_0} = \langle 40, 0, 0\rangle$ and acceleration $\vec{a} = \langle \sqrt{2}, -\sqrt{2}, -10\rangle$. Antidifferentiating, we get $\vec{v} = \langle \sqrt{2}t, -\sqrt{2}t, -10t\rangle + \vec{C}$. Solving $\vec{v}(0) = \vec{v_0}$ gives $\vec{C} = \vec{v_0}$, so $\vec{v} = \langle 40 + \sqrt{2}t, -\sqrt{2}t, -10t\rangle$. Antidifferentiating again, we obtain $\vec{x} = \langle 40t + (\sqrt{2}/2)t^2, (-\sqrt{2}/2)t^2, -5t^2\rangle + \vec{C}$, with $\vec{C} = \vec{x}(0) = \vec{x_0} = \langle 0, 0, 1000\rangle$. So

$$\vec{x} = \langle 40t + (\sqrt{2}/2)t^2, (-\sqrt{2}/2)t^2, 1000 - 5t^2\rangle.$$

The package lands when $z(t) = 0$, so when $5t^2 = 1000$, so when $t = 10\sqrt{2}$. At this time $\vec{x} = \langle 500\sqrt{2}, -100\sqrt{2}, 0\rangle$. The distance from the package to Trevor is the length of this vector, which is $100\sqrt{52}$.

5 (12 pts). Let $f(x, y) = \frac{y^2x^3}{y^4+x^8}$.

A. What does the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ appear to be if we approach the origin along the x-axis? Along the y-axis? Along any other line $y = mx$? Write your answers in the spaces provided.

Solution:

x-axis: 0 (arising as $\lim_{x\to 0} 0/x^8 = \lim_{x\to 0} 0 = 0$)

y-axis: 0 (arising as $\lim_{y\to 0} 0/y^4 = \lim_{y\to 0} 0 = 0$)

$y = mx$: 0 (arising as $\lim_{x\to 0}(m^2x^5)/(m^4x^4 + x^8) = \lim_{x\to 0}(m^2x)/(m^4 + x^4) = 0/m^4 = 0$)

B. Can you conclude that the limit exists, or that it does not exist, or neither? Why?

Solution: Neither; it seems that the limit might exist, and that it might be 0, but we can't say for sure because the limit could differ or fail to exist entirely if we approached the origin along some path other than the lines above.

6 (20 pts). The figure below shows the graph of the parametric plane curve

$$\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t(t-1)^2, t^2(t-1)\rangle = \langle t^3 - 2t^2 + t, t^3 - t)\rangle.$$

A. On the graph, clearly mark all points where $x'(t) = 0$ and all points where $y'(t) = 0$. (You do not need to give their coordinates.)

Solution: [I won't draw the points on the graph, but I will explain. First, $x'(t) = 0$ when the curve has zero velocity in the x-direction — that is, when the curve isn't moving left or right. This occurs at the right-most point in the lower-right quadrant and at the origin. Similarly, $y'(t) = 0$ when the curve has zero velocity in the y-direction; this occurs at the bottom-most point in the lower-right quadrant and at the origin. Notice that the curve passes through the origin at two distinct times — $t = 0$ and $t = 1$ — and that $x'(1) = 0$ and $y'(0) = 0$. The two derivatives do not vanish at the same time; they just happen to vanish at the same place.]

B. Compute the speed v, unit tangent vector $\vec{T}$, curvature κ, and unit normal vector $\vec{N}$ at time $t = 1/2$. Write your final answers in the spaces provided, and draw $\vec{T}$ and $\vec{N}$ on the graph.

Solution:

$v = \sqrt{2}/4$

$\vec{T} = \langle -1/\sqrt{2}, -1/\sqrt{2}\rangle$

$\kappa = 16\sqrt{2}$

$\vec{N} = \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$

[One can find these answers using the formulae $\vec{v} = \vec{r}'$, $v = |\vec{v}|$, $\vec{T} = \vec{v}/v$, and

$$\kappa\vec{N} = \frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} = \frac{1}{v}\frac{d\vec{T}}{dt}.$$

That is, κ is the length of $(1/v)d\vec{T}/dt$ and $\vec{N}$ is $(1/v\kappa)d\vec{T}/dt$. I leave it to you to draw the vectors on the graph. Notice that the vectors are length 1, while the width of the graph is only 1/2, so you should draw them to be very large.]

7 (12 pts). In this problem you will find the point on the surface $x^2y^2z = 4$ (with $x, y, z > 0$) that is closest to the origin.

A. What function are you going to minimize? (If you do not know then you may ask me and I will tell you a function so that you can attempt parts B and C.)

Solution: [By the way, this is 13.5 # 33.] We will minimize the square of the distance between the origin and (x, y, z), which is given by $f(x, y) = x^2 + y^2 + 16x^{-4}y^{-4}$. Minimizing the square of the distance is tantamount to minimizing the distance itself.

B. On what region R are you minimizing this function? Is R a closed, bounded region (the sort that we like to minimize on)? How do you know that you will find a minimum?

Solution: We are minimizing on the region where $x, y > 0$; this is neither closed nor bounded. (In the book's parlance, it does not consist of points on and inside a simple, closed curve.) However, we know that for x or y near 0 the third term in f gets arbitrarily large, and that for x and y large the other two terms get arbitrarily large. Therefore f gets large as we approach any part of the boundary of the region. So if we find only one critical point in the interior of the region then this point must be a minimum. In part C we will find only one critical point.

C. Find the minimum. That is, what is the point on the surface in the first octant that is closest to the origin?

Solution: Setting the partials equal to 0, we have $f_x = 2x - 64x^{-5}y^{-4} = (2x^6y^4 - 64)/(x^5y^4) = 0 \Rightarrow x^6y^4 = 32$ and $f_y = 2y - 64y^{-5}x^{-4} = (2y^6x^4 - 64)/(y^5x^4) = 0 \Rightarrow y^6x^4 = 32$. So $x^6y^4 = y^6x^4$, so $x^2 = y^2$ and so $x = y$ since we are dealing only with positive x, y. Substituting back, we have $32 = x^6y^4 = x^{10} \Rightarrow x = \sqrt{2} = y$. The value of $f(x, y)$ at this critical point is $f(\sqrt{2}, \sqrt{2}) = 5$. There are no other critical points, since the partials are defined whenever $x, y > 0$. So $f(x, y)$ achieves the minimum value 5 at the point $(\sqrt{2}, \sqrt{2})$. This means that the point $(\sqrt{2}, \sqrt{2}, 1)$, which is $\sqrt{5}$ units from the origin, is the point on the surface closest to the origin.

8 (8 pts). Mathematician Emmy Noether (1882 - 1935) is driving her motorcycle due south through scenic Germany. The surface of the land around her is described by the graph of

$$z = e^{x^2 - y^2},$$

with the positive x- and y-axes pointing east and north, as usual. When she reaches the point $(0, 1, 1/e)$, she suddenly turns off gravity (she's that good) and goes flying off tangentially to the surface. Describe her flight trajectory as a parametrized line.

Solution: First, $\partial z/\partial y = e^{x^2-y^2}(-2y)$, which equals $-2/e$ at $(0, 1, 1/e)$. This is the slope (meaning the change in z over the change in y) of the tangential line. So $\langle 0, 1, -2/e\rangle$ is a vector parallel to that line. The line is therefore

$$\vec{x}(t) = \langle 0, 1, 1/e\rangle + t\langle 0, 1, -2/e\rangle.$$

[Remark: Since Emmy is going south, the vector we found points backwards along her flight trajectory. It is arguably better to use the opposite vector, $\langle 0, -1, 2/e\rangle$ to describe her flight direction, but this is not strictly necessary.]

9 (8 pts). On a recent trip to suburban Madagascar I met a UNC professor who claimed that her favorite function $f(x, y)$ had partial derivatives

$$f_x = \frac{-y}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{x}{\sqrt{x^2 + y^2}}.$$

I was skeptical that such a function $f(x, y)$ could even exist. She conceded that $f(x, y)$ was not defined at the origin, but asserted that it was defined everywhere else.

What do you think: Does such a function exist? Explain your evidence for or against it. (You are not required to find the function.)

Solution: If the function exists, then f_{xy} must equal f_{yx} everywhere they are continuous. But

$$f_{xy} = \frac{-(x^2 + y^2)^{1/2} + y^2(x^2 + y^2)^{-1/2}}{x^2 + y^2} = \frac{-x^2}{(x^2 + y^2)^{3/2}},$$

while

$$f_{yx} = \frac{(x^2 + y^2)^{1/2} - x^2(x^2 + y^2)^{-1/2}}{x^2 + y^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}.$$

They are not equal, so $f(x, y)$ cannot exist. [Remark: If you remove the square root signs from the statement of the problem, then it does turn out that $f_{xy} = f_{yx}$. However, this is not enough to conclude that $f(x, y)$ exists; in fact, such an $f(x, y)$ does not exist. If you try to find $f(x, y)$, you might understand why. We will return to this question later in the semester.]