The flow of a vector field.

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field in the plane[1] Associated to $\mathbf{F}$ is its **flow** which, for each time t is a transformation

$$\mathbf{f}_t(x,y) \;=\; (u_t(x,y), v_t(x,y))$$

and which is characterized by the requirements that

$$\mathbf{f}_0(x,y) \;=\; (x,y) \tag{1}$$

and

$$\frac{d}{dt}\mathbf{f}_t(x,y) \;=\; \mathbf{F}(\mathbf{f}_t(x,y)). \tag{2}$$

That is, for each (x,y), $t \mapsto \mathbf{f}_t(x,y)$ is a path whose velocity at time t is the vector that $\mathbf{F}$ assigns to $\mathbf{f}_t(x,y)$.

Example. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Draw a picture of $\mathbf{F}$. Note that

$$\mathbf{f}_t(x,y) \;=\; (x\cos t - y\sin t, x\sin t + y\cos t).$$

That is, $\mathbf{f}_t$ is counterclockwise rotation of $\mathbf{R}^2$ through an angle of t radians.

As a direct consequence of (1) and (2) above find that

$$u_t(x,y) \;=\; x + tP(x,y) + t^2 r_t(x,y) \quad \text{and} \quad v_t(x,y) \;=\; y + tQ(x,y) + t^2 s_t(x,y).$$

It follows that

$$\begin{aligned}
J_{\mathbf{f}_t}(x,y) &= \mathbf{det}\begin{bmatrix} \frac{\partial u_t}{\partial x} & \frac{\partial u_t}{\partial y} \\ \frac{\partial v_t}{\partial x} & \frac{\partial v_t}{\partial y} \end{bmatrix}(x,y) \\
&= \begin{bmatrix} 1 + tP_x(x,y) + t^2\frac{\partial r_t}{\partial x}(x,y) & tP_y(x,y) + t^2\frac{\partial r_t}{\partial y}(x,y) \\ tQ_x(x,y) + t^2\frac{\partial s_t}{\partial x}(x,y) & 1 + tQ_y(x,y) + t^2\frac{\partial s_t}{\partial y}(x,y) \end{bmatrix} \\
&= 1 \;+\; t(P_x + Q_y)(x,y) \;+\; t^2 z_t(x,y).
\end{aligned}$$

This implies that

$$\frac{d}{dt}J_{\mathbf{f}_t}(x,y)\Big|_{t=0} \;=\; \mathbf{div}\,\mathbf{F}(x,y)$$

where we have set

$$\mathbf{div}\,\mathbf{F} \;=\; P_x + Q_y.$$

This yields the following basic formula:

Theorem. Suppose R is a bounded region in the domain of $\mathbf{F}$. Then

$$\frac{d}{dt}\mathbf{area}\,\mathbf{f}_t(R)\Big|_{t=0} \;=\; \iint_R \mathbf{div}\,\mathbf{F}\,dA.$$

Analogous formulae hold in any dimension.

[1] Everything we are about to do here directly generalized to any number of dimensions.