Relations and functions.

A **relation** is a set of ordered pairs.
Let r be a relation. The **domain** of r, denoted by

$$\mathbf{dmn}\, r,$$

is the set $\{x : \text{for some } y,\ (x, y) \in r\}$, and the the **range** of r, denoted by

$$\mathbf{rng}\, r,$$

is the set $\{y : \text{for some } x,\ (x, y) \in r\}$. Suppose A is a set. The **restriction of** r **to** A, denoted by

$$r|A,$$

is the relation $\{(x, y) \in r : x \in A\}$. r **of** A, denoted by

$$r[A],$$

is the set $\{y : \text{for some } x,\ x \in A \text{ and } (x, y) \in r\}$. The **inverse** of r, denoted by

$$r^{-1},$$

is the relation $\{(y, x) : (x, y) \in r\}$.
If r and s are relations the **composition of** s **with** r, denoted by

$$s \circ r,$$

is the relation $\{(x, z) : \text{for some } y,\ (x, y) \in r \text{ and } (y, z) \in s\}$. The operation of composition of relations is easily seen to be associative which is to say that

$$t \circ (s \circ r) = (t \circ s) \circ r$$

whenever r, s and t are relations.
We say the relation f is a **function** if

$$(x, y_1) \in f \text{ and } (x, y_2) \in f \ \Rightarrow\ y_1 = y_2.$$

If f is a function and A is a set then $f|A$ is a function. If f is a function and $x \in \mathbf{dmn}\, f$ we let

$$f(x)$$

be the unique y such that $(x, y) \in f$. We say a function f is **one-to-one** if f^{-1} is a function. If f and g are functions then the composition $g \circ f$ is a function, its domain is

$$f^{-1}[\mathbf{dmn}\, g]$$

and

$$g \circ f(x) = g(f(x)) \quad \text{whenever } x \in f^{-1}[\mathbf{dmn}\, g].$$

We frequently write

$$f : X \to Y$$

if f is a function, the domain of f is the set X, Y is a set and the range of f is a subset of Y. Note that if f is a function and B is a set then

$$f^{-1}[B] = \{x : f(x) \in B\}.$$

Example. Let

$$\text{son} = \{(x,y) : x \text{ is the son of } y\}.$$

Repeat this for any other human relationship you care to. Let M be the set of males and let F be the set of females. Play around with this.

We say a relation r is **numerical** if each component of each of its members is a real number. Equivalently, a function is numerical if each ordered pair in it is an ordered pair of real numbers.

Example. For each nonnegative integer n we define the numerical function

$$p_n : \mathbf{R} \to \mathbf{R}$$

by induction by requiring that

$$p_0 = \{(x,1) : x \in \mathbf{R}\}$$

and that and

$$p_{n+1} = \{(x,xy) : (x,y) \in p_n\}.$$

For a real number x, x^n is just $p_n(x)$; thus the "p" stands for "power". Check that

$$p_m \circ p_n = p_{mn}.$$

As an example of this notation, note that

$$(p_2|[0,\infty))^{-1}$$

is the square root function.