The equality of mixed partial derivatives.

Suppose $A \subset \mathbf{R}^2$ and

$$f : A \to \mathbf{R}.$$

Suppose (a, b) is an interior point of A near which the partial derivatives

$$\frac{\partial f}{\partial x}, \; \frac{\partial f}{\partial y}, \; \frac{\partial^2 f}{\partial x \partial y}, \; \frac{\partial^2 f}{\partial y \partial x}$$

exist.

Let

$$S(h, k) \;=\; f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).$$

Then

$$(1) \qquad \lim_{(h,k)\to 0} \frac{S(h,k)}{hk} \;=\; \frac{\partial^2 f}{\partial x \partial y}(a, b)$$

if $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (a, b) and

$$(2) \qquad \lim_{(h,k)\to 0} \frac{S(h,k)}{hk} \;=\; \frac{\partial^2 f}{\partial y \partial x}(a, b)$$

if $\frac{\partial^2 f}{\partial y \partial x}$ is continuous at (a, b). In particular, if *both* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ are continuous at (a, b) then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) \;=\; \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Here is the proof of this fundamental fact. Let

$$A(x, y) \;=\; f(x, y) - f(x, b)$$

and note that

$$\frac{\partial A}{\partial x}(x, y) \;=\; \frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial x}(x, b).$$

Using the Mean Value Theorem twice, we find that

$$\begin{aligned} S(h, k) \;&=\; A(a+h, b+k) - A(a, b+k) \\ &=\; h\frac{\partial A}{\partial x}(\xi, b+k) \\ &=\; h\Big(\frac{\partial f}{\partial x}(\xi, b+k) - \frac{\partial f}{\partial x}(\xi, b)\Big) \\ &=\; hk\frac{\partial^2 f}{\partial y \partial x}(\xi, \eta), \end{aligned}$$

where ξ and η are such that $0 < |\xi - a| < |h|$ and $0 < |\eta - b| < |k|$, respectively. Thus if $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (a, b) then (1) holds.

To prove (2) we let

$$B(x, y) \;=\; f(x, y) - f(a, y),$$

note that

$$\frac{\partial B}{\partial y}(x, y) \;=\; \frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(a, y)$$

note that $S(h, k) = B(a+h, b+k) - B(a+h, b)$ and proceed as above.