Math 104-01, Spring 2006, Exam 2

Instructions: This is an unlimited-time, open-book take-home exam — sort of like a homework assignment on which you are not allowed to collaborate. The exam is due at the start of class on Wednesday, 29 March 2006. I anticipate that it will take longer than one day to complete — about as long as a homework assignment? — and you may find it helpful to revisit a problem over several days. So I recommend that you get started as soon as possible.

Your solutions should be polished (concise, neat, and well-written, employing complete sentences with punctuation) and self-explanatory. Submit them in a single stapled packet, presented in the order they were assigned. Always show enough work so that I can follow your solution, but do not show scratch work (false starts, circuitous reasoning, etc.). Quantitative answers should always be exact and simplified.

Partial credit is often awarded. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Exam grades will be curved, as usual.

Write and sign the honor pledge on your packet of solutions. Here are the rules:

- You may freely consult all of this class' material: the textbook, your class notes, your old homework, your old exam, and the class web site. If you missed a lecture and need to copy someone else's class notes, do so before beginning the exam.
- You may talk to me in private. You may ask clarifying questions for free. If you're really stuck on a problem, then you may ask for a hint, which will cost you some points. The opportunity to ask questions is another reason to get started early.
- You may not cite theorems from later parts of the book that we have not studied. Your solutions should make use of techniques covered thus far.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may not use a calculator or computer, except to view the class web site.
- You may not discuss the exam in any way (spoken, written, pantomime, etc.) with anyone but me. During the exam you will inevitably see your classmates around campus. Please refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" (If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.)

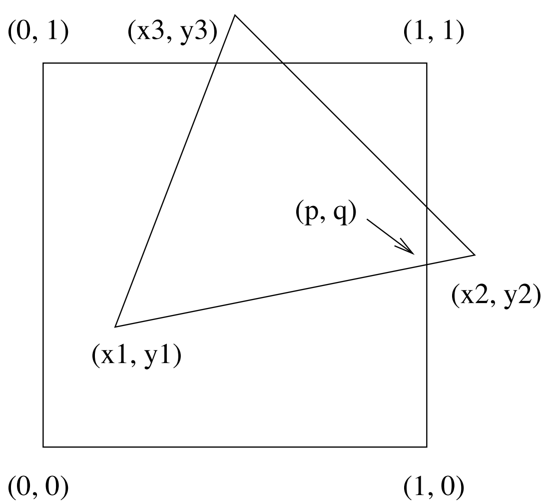
If you have any questions about the exam or its rules, then ask for clarification.

1. In our discussion of 3D graphics, we studied how to rotate, translate, and project triangles in $\mathbb{R}^3$ into a plane. We did not discuss how triangles in a plane (let's say it's just $\mathbb{R}^2$) are then drawn. One problem is that in practice we don't really have an infinitely large plane on which to draw, but just a finite, usually rectangular computer screen or movie screen. For simplicity, suppose that our screen is the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ sitting in $\mathbb{R}^2$. A projected triangle might lie in the screen, in which case we draw it, or it might lie entirely outside the screen, in which case we don't draw it. Then again, the triangle might lie partly in the screen and partly outside it; we only want to draw the part of the triangle that lies in the screen.

Consider the figure below. The big triangle has one vertex (x_1, y_1) in the screen, and its two other vertices (x_2, y_2) and (x_3, y_3) lie outside the screen. We want to draw the intersection of the triangle with the screen, which is a pentagon. One vertex of the pentagon is (x_1, y_1), and the other four are currently unknown. To find them, we need to intersect the sides of the triangle with the sides of the screen.

A. Give a formula for the intersection point (p, q) in terms of (x_1, y_1) and (x_2, y_2).

B. In general, if I give you any two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ (not necessarily arranged as in the figure below — they can now be anywhere), then the line segment between the two might not intersect the screen at all, in which case the computation from Part A is not really useful. Given any two points (x_1, y_1) and (x_2, y_2), find a specific criterion that detects whether the line segment between them intersects the line segment between $(1, 0)$ and $(1, 1)$.



2. For any vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ in $\mathbb{R}^2$, define

$$\langle \vec{v}, \vec{w} \rangle = v_1 w_1 - v_2 w_2.$$

This is similar to the dot product, but with a minus sign. Define $||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, as one usually defines a norm from an inner product.

A. It turns out that this $\langle \vec{v}, \vec{w} \rangle$ does not define an inner product on $\mathbb{R}^2$. Which parts of the definition of inner product does it satisfy, and which parts does it not satisfy?

B. It also turns out that $||\vec{v}||$ doesn't define a norm; for one thing, it's not even defined everywhere. For which vectors $\vec{v} \in \mathbb{R}^2$ is $||\vec{v}||$ defined? For which $\vec{v}$ is it 0? For which $\vec{v}$ is it 1? Answer these questions both in words/equations and in a detailed sketch of $\mathbb{R}^2$.

C. Terminology: The *determinant* of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be the quantity $ad - bc$. A matrix is said to be *special* if it has determinant 1.

In Exercise 2.3 #13 you showed that all 2×2 special orthogonal matrices are of the form

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some number θ. (The other form in the exercise has determinant -1, as you can check; I'm not interested in that.) In Exercise 2.1 #9a you also proved that these matrices preserve the standard norm on $\mathbb{R}^2$, meaning that the norm of $A_\theta\vec{v}$ equals the norm of $\vec{v}$.

Now I want you to do the same thing for the fake norm $||\vec{v}||$ defined above. That is, give a formula for matrices B_θ such that the determinant of B_θ is 1 and $||B_\theta\vec{v}||^2 = ||\vec{v}||^2$ for all $\vec{v} \in \mathbb{R}^2$. Explicitly show that your B_θ satisfies both of these properties. (Hint: Instead of using cos and sin, use cosh and sinh. Recall that these are defined as $\cosh\theta = (e^\theta + e^{-\theta})/2$ and $\sinh\theta = (e^\theta - e^{-\theta})/2$. They satisfy the identities such as $\cosh^2\theta - \sinh^2\theta = 1$, $\cosh(-\theta) - \cosh\theta$, and $\sinh(-\theta) = -\sinh\theta$. Be careful about sign errors.)

3. Let A be any $n \times n$ skew-symmetric matrix (which means that $A^\top = -A$, recall).

A. Prove that the kth power of A is symmetric if k is even and skew-symmetric if k is odd.

B. For any odd number k, prove that $\vec{v} \cdot (A^k\vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^n$.

C. Prove that if $n = 3$ then A must be singular. (In fact, A must be singular for any odd n, but you don't have to prove that.)

4. In Section 3.6 (say, Example 1d) we saw several examples of vector spaces of functions. Now we will generalize all of them. Let X be any set and V any vector space (not necessarily finite-dimensional). Let F be the set of all functions from X to V. (All you know about a function $f \in F$ is that for each element $x \in X$ it produces a vector $f(x) \in V$.)

A. Show that F is a vector space under the operations of function addition and scalar multiplication. (We say that "F inherits a vector space structure from V".)

B. Let x be any particular element of X and $F_x \subseteq F$ the set of functions f such that $f(x) = 0$. Show that F_x is a subspace of F.

C. Suppose that V possesses an inner product $\langle\,,\rangle$. Suppose also that the set X is finite; say $X = \{x_1, \ldots, x_k\}$. Define

$$\langle\langle f, g\rangle\rangle = \sum_{i=1}^{k} \langle f(x_i), g(x_i)\rangle.$$

Is this $\langle\langle\,,\rangle\rangle$ an inner product on F? Prove or disprove it.

D. Again suppose that $X = \{x_1, \ldots, x_k\}$ is a finite set. Also suppose that V is of finite dimension n. Determine the dimension of F, and find a basis for it. (Suggestion: You might want to try a few small examples first.)

5. In $\mathbb{R}^4$, let $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -1 \\ -2 \\ 4 \\ 2 \end{bmatrix}$. Find a basis for the orthogonal complement of the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.

6. You may already know the formulae

$$1 + 2 + \cdots + n = \frac{1}{2}n^2 + \frac{1}{2}n, \qquad 1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

I want to find a similar formula for the sum of fourth powers. Fortunately, there is a theorem that says that, for any positive integer k (in my case, $k = 4$) there exist constants a_1, a_2, $\ldots$, a_{k+1} such that for all $n \geq 1$,

$$\sum_{i=1}^{n} i^k = a_{k+1}n^{k+1} + a_k n^k + \cdots + a_1 n.$$

You do not have to prove this theorem, but do explain how you would use linear algebra to find the promised formula for $k = 4$. That is, write a system of linear equations whose solution gives the coefficients a_i in the $k = 4$ formula. (You do not have to solve the linear system.)