Math 104-01, Spring 2006, Exam 3 Solutions

Instructions: This is an unlimited-time, open-book take-home exam, due at the start of class on Monday, 24 April 2006. The requirements are identical to those of our previous exam; I have reprinted them below. Since the exam takes place over the weekend, you will have to ask questions via e-mail, not in person; I will check my e-mail frequently.

Your solutions should be polished (concise, neat, and well-written, employing complete sentences with punctuation) and self-explanatory. Submit them in a single stapled packet, presented in the order they were assigned. Always show enough work so that I can follow your solution, but do not show scratch work (false starts, circuitous reasoning, etc.). Quantitative answers should always be exact and simplified.

Partial credit is often awarded. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Exam grades will be curved, as usual.

Write and sign the honor pledge on your packet of solutions. Here are the rules:

- You may freely consult all of this class' material: the textbook, your class notes, your old homework, your old exam, and the class web site. If you missed a lecture and need to copy someone else's class notes, do so before beginning the exam.
- You may talk to me in private. You may ask clarifying questions for free. If you're really stuck on a problem, then you may ask for a hint, which will cost you some points. The opportunity to ask questions is another reason to get started early.
- You may not cite theorems from later parts of the book that we have not studied. Your solutions should make use of techniques covered thus far.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may not use a calculator or computer, except to view the class web site.
- You may not discuss the exam in any way (spoken, written, pantomime, etc.) with anyone but me. During the exam you will inevitably see your classmates around campus. Please refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" (If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.)

If you have any questions about the exam or its rules, then ask for clarification.

1. Let A be an $n \times n$ symmetric matrix that is also *positive-definite*, meaning that $\vec{x}^\top A\vec{x} > 0$ for all nonzero column vectors $\vec{x} \in \mathbb{R}^n$.

A. Prove that A is nonsingular.

Solution: Assume that the null space of A contains some nonzero vector $\vec{x}$. Then

$$\vec{x}^\top A\vec{x} = \vec{x}^\top \vec{0} = 0,$$

contradicting the assumption of positive-definiteness. This proves (by contradiction) that the null space contains only the $\vec{0}$ vector. So A is nonsingular.

B. It is not difficult to check (but you do not have to) that A defines an inner product on $\mathbb{R}^n$ by the formula

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^\top A\vec{y}.$$

Here we are implicitly working with respect to the standard basis $\mathcal{E} = \{\vec{e_1}, \ldots, \vec{e_n}\}$, as we did for the whole first half of the course. Nowadays we know that we should really label the column vectors and the matrix A with the basis being used, writing $\vec{x}_\mathcal{E}$ instead of $\vec{x}$, $\vec{y}_\mathcal{E}$ instead of $\vec{y}$, and $A_\mathcal{E}$ instead of A — since if we work with respect to a different basis $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ then this same inner product is probably represented by a different matrix, $A_\mathcal{B}$.

Derive the change-of-basis formula for inner products. That is, derive a formula for the matrix $A_\mathcal{B}$ such that $\vec{x}_\mathcal{B}^\top A_\mathcal{B}\vec{y}_\mathcal{B} = \vec{x}_\mathcal{E}^\top A_\mathcal{E}\vec{y}_\mathcal{E}$. (Hint: You may wish to emulate the derivation of the change of basis formula for linear transformations. In your answer, make sure to be clear what your change-of-basis matrix P is.)

Solution: Let P be the $n \times n$ matrix whose columns are the coordinate vectors of $\vec{v_1}, \ldots, \vec{v_n}$ with respect to the standard basis. Then for any vector $\vec{v} \in \mathbb{R}^n$ with coordinates $\vec{x}_\mathcal{E}$, $\vec{x}_\mathcal{B}$ with respect to the two bases, we already know that

$$\vec{x}_\mathcal{E} = P\vec{x}_\mathcal{B}.$$

Let $\vec{w}$ denote another vector with coordinates $\vec{y}_\mathcal{E}$, $\vec{y}_\mathcal{B}$. We want to find $A_\mathcal{B}$ such that

$$\vec{x}_\mathcal{B}^\top A_\mathcal{B}\vec{y}_\mathcal{B} = \vec{x}_\mathcal{E}^\top A_\mathcal{E}\vec{y}_\mathcal{E}.$$

Working backwards, we replace $\vec{x}_\mathcal{E}$ with $P\vec{x}_\mathcal{B}$ and $\vec{y}_\mathcal{E}$ with $P\vec{y}_\mathcal{B}$ to obtain

$$\vec{x}_\mathcal{B}^\top A_\mathcal{B}\vec{y}_\mathcal{B} = (P\vec{x}_\mathcal{B})^\top A_\mathcal{E}(P\vec{y}_\mathcal{B}) = \vec{x}_\mathcal{B}^\top (P^\top A_\mathcal{E} P)\vec{y}_\mathcal{B}.$$

Since this must hold for all vectors $\vec{x}_\mathcal{B}$ and $\vec{y}_\mathcal{B}$, we conclude that $A_\mathcal{B} = P^\top A_\mathcal{E} P$.

2. Let A be any $n \times n$ matrix and

$$p(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_n\lambda^n$$

its characteristic polynomial. Let M denote the matrix obtained by plugging A into $p(\lambda)$; to be precise,

$$M = p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n.$$

A. Prove that $M\vec{v} = \vec{0}$ for each eigenvector $\vec{v} \in \mathbb{R}^n$ of A.

Solution: If λ is the eigenvalue for the eigenvector $\vec{v}$, then

$$\begin{aligned} M\vec{v} &= (c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n)\vec{v} \\ &= c_0 I\vec{v} + c_1 A\vec{v} + c_2 A^2\vec{v} + \cdots + c_n A^n\vec{v} \\ &= c_0\vec{v} + c_1\lambda\vec{v} + c_2\lambda^2\vec{v} + \cdots + c_n\lambda^n\vec{v} \\ &= (c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_n\lambda^n)\vec{v} \\ &= 0\vec{v} \\ &= \vec{0}. \end{aligned}$$

B. Now assume that A possesses n linearly independent eigenvectors $\vec{v_1}, \ldots, \vec{v_n}$. (In other words, A is diagonalizable; however, you don't need to know anything beyond Section 6.1 to solve this problem.) Prove that M is the zero matrix. (Therefore the matrix A is a root of its own characteristic polynomial.)

Solution: From Part A we know that $\vec{v_1}, \ldots, \vec{v_n}$ are all in the null space of M. Since these are linearly independent, the null space of M must be n-dimensional, namely all of $\mathbb{R}^n$. Therefore M is the zero matrix.

3. In geology, the San Andreas fault system in California forms part of the boundary between the Pacific and North American tectonic plates. The relative motion of these two plates is complicated; they are sliding past each other while also moving toward each other slightly (and rotating, but forget about that). As a result, the rocks along the fault zone are deformed both by shearing (in a direction parallel to the fault line) and compression (perpendicular to the fault) — a combination called *transpression*. Structural geologists often model such deformation of rock using linear transformations.

A. Let $A = \begin{bmatrix} 1 & 0.3 \\ 0 & 0.9 \end{bmatrix}$; this matrix defines a linear transformation $\mu_A : \mathbb{R}^2 \to \mathbb{R}^2$ in the usual way, with respect to the standard basis. Let S be the square (made of rock) in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. How is this square transformed by A? Draw a detailed sketch.

Solution: [Transform each vertex by the matrix, and draw in the parallelogram they form. Notice that it has been compressed in the y-direction and sheared in the x-direction.]

B. By convention, the positive x-axis points east and the positive y-axis points north. The matrix A, then, represents transpression along a fault that runs east-west. In real life, the San Andreas fault runs northwest-southeast (roughly), so we need to perform a rotational change-of-basis on A to model deformation along this fault.

Using change-of-basis on A, find a matrix B that models transpression along the San Andreas fault. Simplify B as much as possible, after deriving it. (To check your answer, you might test it on a vector that points along the fault; what should B do to such a vector?)

Solution: To transpress along the San Andreas fault, we want to rotate the plane by $-3\pi/4$, then transpress it by A, and then rotate it back. So we use the matrix

$$P = \begin{bmatrix} \cos(-3\pi/4) & -\sin(-3\pi/4) \\ \sin(-3\pi/4) & \cos(-3\pi/4) \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}.$$

The desired matrix is

$$B = P^{-1}AP = \begin{bmatrix} 1.1 & 0.1 \\ -0.2 & 0.8 \end{bmatrix}.$$

You can check that $B \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; this is a vector pointing along the fault, and B shouldn't change it. Also, $B \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -0.6 \end{bmatrix}$; this shows that a vector pointing into the Pacific plate gets transpressed in a vaguely northwest direction, as desired.

C. Parts A and B dealt with deformation in two dimensions, but of course it's all really happening in three dimensions. So let's enlarge A to the 3×3 matrix

$$C = \begin{bmatrix} 1 & 0.3 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & k \end{bmatrix}.$$

When a rock deforms, its volume should remain constant (unless there are chemical changes occurring). Assuming that the deformation C preserves volume, what is k?

Solution: The determinant measures the volume change; it must be 1. It follows that k must be $10/9$. [This means that some of the rock is moving out of the plane, in the third dimension.]

4. Find the determinant of

$$\begin{bmatrix} 1 & -3 & 2 & 0 & -1 & 0 \\ 2 & \pi & 0 & -1 & 3 & 1 \\ -5 & 17000 & 4 & -1 & -1 & 2 \\ 2 & \pi & 0 & -1 & 3 & 1 \\ -1 & -1 & -1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 1 & 2 & 0 \end{bmatrix}.$$

Fully explain your answer. (Hint: There are several ways to do this; one is extremely easy.)

Solution: The second and fourth rows are equal — linearly dependent in the most flagrant manner! Therefore the matrix is singular and its determinant is 0.