MATH 104, SPRING 2006, ASSIGNMENT 1

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Learning how to write rigorous, coherent mathematical arguments is a major component of this class. There are two main points about writing to keep in mind:

- Each solution should be clear and concise, consisting of a sequence of complete English sentences, usually with some words written in mathematical notation. Each sentence should have a verb and punctuation. (The most commonly used verb in math is probably "equals", and it is usually written "=".) After you write a solution, read it aloud to make sure it's sensible.
- How much work should you show? The rule of thumb is to write as if your reader is another student in the class (who doesn't know how to do the problem). Your solution should be self-explanatory, noting any new material or novel strategy that you employ.

I'm giving you a few example solutions so that you can get an idea of how yours might look. Your solutions do not have to look like mine to be correct. You certainly do not have to type them, and you should probably use more drawings. [I'm also putting in extra editorial comments in braces.]

Revise your solutions for resubmission by Monday. During the revision, you must have at least one other student in the class read your solutions and offer comments.

4. Let $\vec{x} = \vec{x_0} + t\vec{v}$ and $\vec{y} = \vec{y_0} + s\vec{w}$ be two parametrizations of the line ℓ in $\mathbb{R}^n$.

A. Since $\vec{y_0}$ lies on ℓ, and ℓ consists exactly of those points of the form $\vec{x_0} + t\vec{v}$ for varying t, it follows that $\vec{y_0} = \vec{x_0} + t_0\vec{v}$ for some t_0. A symmetric argument shows that $\vec{x_0} = \vec{y_0} + s_0\vec{w}$ for some s_0.

B. Let $\vec{z_0}$ be some point on ℓ other than $\vec{x_0}$ and $\vec{y_0}$. So

$$\vec{z_0} = \vec{x_0} + t_1\vec{v} = \vec{y_0} + s_1\vec{w}$$

for some t_1, s_1. Thus

$$\begin{aligned} & \vec{x_0} + t_1 v = \vec{y_0} + s_1\vec{w} = \vec{x_0} + t_0\vec{v} + s_1\vec{w} \\ \Rightarrow \quad & (t_1 - t_0)\vec{v} = s_1\vec{w}. \end{aligned}$$

If t_1 and t_0 were equal, then $\vec{z_0} = \vec{x_0} + t_1\vec{v}$ and $\vec{y_0} = \vec{x_0} + t_0\vec{v}$ would be equal; however, we chose $\vec{z_0}$ to be distinct from $\vec{y_0}$, so we know that $t_1 \neq t_0$. Therefore $t_1 - t_0 \neq 0$, so

$$\vec{v} = \frac{s_1}{t_1 - t_0}\vec{w}.$$

This means that $\vec{v}$ and $\vec{w}$ are parallel.

[I have set off the more complicated globs of math notation on their own lines. Notice, however, that even a big math glob is grammatically part of its surrounding sentence. By the way, the "$\Rightarrow$" symbol is shorthand for "implies that" or "which implies that". I often use this on the in lecture.]

5B. Let $P = (1, 1, 1)$, $Q = (-2, 1, 2)$, and $\vec{v} = \langle 1, 3, 1 \rangle$. The plane through Q, parallel to $\vec{QP} = \langle 3, 0, -1 \rangle$ and $\vec{v}$, consists of points $\vec{x}$ of the form

$$\begin{aligned} \vec{x} &= \vec{Q} + s\vec{v} + t\vec{QP} \\ &= \langle -2, 1, 2 \rangle + s\langle 1, 3, 1 \rangle + t\langle 3, 0, -1 \rangle \\ &= \langle -2, 1, 2 \rangle + \langle s, 3s, s \rangle + \langle 3t, 0, -t \rangle \\ &= \langle -2 + s + 3t, 1 + 3s, 2 + s - t \rangle. \end{aligned}$$

[First, I am maintaining some distinction between the point $Q = (-2, 1, 2)$ and the vector $\vec{Q} = \langle -2, 1, 2 \rangle$, but not very seriously. Soon I will abandon this entirely. Second, the end of the solution is a sequence of equations; it is too long and complicated to fit on one line, so I've written its successive "terms" on separate lines, with the equals signs lined up, to help the reader. The first "=" is pronounced "equals", while the later ones are pronounced "which equals". (Your book does a similar thing on page 274, for example.) Lastly, I have chosen to show the algebraic steps in close detail, because algebraic manipulation of vectors is new material for us. Later in the semester, I would feel free not to show some of these steps.]

7C. The vector $(0, 1, 2)$ is a linear combination of $(1, 0, 1)$ and $(-2, 1, 0)$:

$$2(1, 0, 1) + 1(-2, 1, 0) = (2, 0, 2) + (-2, 1, 0) = (0, 1, 2).$$

[I could omit the words "The vector", but then I would be beginning a sentence with mathematical notation. That is frowned upon, because it makes things very hard to read. So I threw in some plain English words there, purely for style. Even with these, the solution is quite short; writing in complete sentences does not mean that you have to write a lot! Also notice that I'm now using parentheses "$(,)$" instead of angle brackets "$\langle , \rangle$" for vectors; I really don't care.]

17. Let ℓ be the line $\vec{x} = \vec{x_0} + t\vec{v}$ and m the line $\vec{x} = \vec{x_1} + s\vec{u}$.

Assume first that ℓ and m intersect. This means that there is a point $\vec{p}$ on both ℓ and m. So there exist t_0 and s_0 such that

$$\vec{p} = \vec{x_0} + t_0\vec{v} = \vec{x_1} + s_0\vec{u},$$

which implies that

$$\vec{x_0} - \vec{x_1} = s_0\vec{u} + -t_0\vec{v}.$$

We have demonstrated $\vec{x_0} - \vec{x_1}$ as a linear combination of $\vec{u}$ and $\vec{v}$, which means that it is in their span.

For the converse, assume that $\vec{x_0} - \vec{x_1}$ is in the span of $\vec{u}$ and $\vec{v}$. So there exist a, b such that

$$\vec{x_0} - \vec{x_1} = a\vec{u} + b\vec{v},$$

which implies that

$$\vec{x_0} + -b\vec{v} = \vec{x_1} + a\vec{u}.$$

Call this vector $\vec{p}$. Then $\vec{p}$ (regarded as a point) lies on both ℓ and m, so these two lines must intersect.

Therefore the two lines intersect if and only if $\vec{x_0} - \vec{x_1}$ is in the span of $\vec{u}$ and $\vec{v}$.

[We are trying to prove a statement of the form "P if and only if Q". The standard way to do this is to assume P and prove Q, and then assume Q and prove P. (Sometimes you can combine these; you could probably combine them in this problem, but I'm trying to illustrate the method.) I have put each of these steps into its own paragraph, to aid the reader. I also threw on a final sentence summarizing what we've shown. My solution is entirely self-contained; the reader knows what the problem was and how we solved it, without looking it up.]

20. Let $\vec{x} = \langle x_1, \ldots, x_n \rangle$ and $\vec{y} = \langle y_1, \ldots, y_n \rangle$ be vectors in $\mathbb{R}^n$.
A.

$$\begin{aligned} \vec{x} + \vec{y} &= \langle x_1, \ldots, x_n \rangle + \langle y_1, \ldots, y_n \rangle \\ &= \langle x_1 + y_1, \ldots, x_n + y_n \rangle \\ &= \langle y_1 + x_1, \ldots, y_n + x_n \rangle \\ &= \langle y_1, \ldots, y_n \rangle + \langle x_1, \ldots, x_n \rangle \\ &= \vec{y} + \vec{x}. \end{aligned}$$

[In this case my entire solution is a single sequence of equations, with no plain English words. I don't think plain words would help its readability; it's already quite readable. I've included the "Let $\vec{x} = \ldots$" part in order to establish my notation. This problem also asks for a "geometric proof", meaning a diagram and explanation of the geometric meaning.]

21A. Using, in order, properties C, A, D, B, H, G, H, and D of problem 20,

$$\begin{aligned} 0\vec{x} &= \vec{0} + 0\vec{x} \\ &= 0\vec{x} + \vec{0} \\ &= 0\vec{x} + (\vec{x} + -\vec{x}) \\ &= (0\vec{x} + \vec{x}) + -\vec{x} \\ &= (0\vec{x} + 1\vec{x}) + -\vec{x} \\ &= (0 + 1)\vec{x} + -\vec{x} \\ &= 1\vec{x} + -\vec{x} \\ &= \vec{x} + -\vec{x} \\ &= \vec{0}. \end{aligned}$$

[Again it's a string of equations, preceded by a small explanation of which properties I'm using, since that's what this problem is all about. There may be a shorter solution; let me know if you find one (with the same level of rigor). Notice that I did not resort to using coordinates $\vec{x} = \langle x_1, \ldots, x_n \rangle$; I am supposed to use only the abstract properties given in problem 20. The reason becomes clear on page 199.]