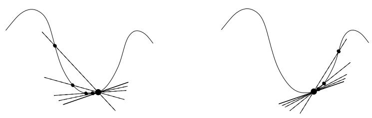
One-Page Summary of Derivatives, by Joshua R. Davis, jdavis@math.wisc.edu, 31 Dec 2003

Differential calculus is motivated by the following question: Given a function $f(x)$, how fast is the function increasing or decreasing at any particular x-value x_0? In other words, what is the "slope" of the graph $y = f(x)$ at the point $(x_0, f(x_0))$? By the slope of the graph at a point we mean the slope of its tangent line, which is the line that best approximates the graph there.

In order to express the slope of the tangent line symbolically, we take a limit of slopes of secant lines, as follows. Let $x_1 = x_0 + h$ be another x-value. The points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ determine a line L with slope $\frac{f(x_1)-f(x_0)}{x_1-x_0}$. Now we let x_1 approach x_0; the line L should approach the tangent line at x_0, and so its slope should approach the slope of the tangent.



Definition. The *derivative* of f at x_0 is $f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0}$, if this limit exists. We say that f is *differentiable at* x_0 if $f'(x_0)$ exists at x_0; f is *differentiable* if it is differentiable at every point in its domain.

At each x where f is differentiable we have a slope $f'(x)$, so we view the derivative as a function f' whose value at x is the slope of f there. If we differentiate this function f', then we get $(f')'$, or f'', which we call the *second derivative* of f. Differentiating further, we get additional derivatives f''', f'''', etc. This notation becomes cumbersome, so we write $f^{(i)}$ for the ith derivative. For example, $f^{(1)} = f'$, and $f^{(0)} = f$.

A common alternative notation for the derivative is $\frac{d}{dx}f$. Here $\frac{d}{dx}$ means "the derivative, with respect to x, of". In notation, we often pretend that $\frac{d}{dx}$ is a fraction (although it is not); for an equation $y = f(x)$, these are all common notations for the derivative:

$$f' = \frac{d}{dx}(f) = \frac{d}{dx}f = \frac{df}{dx} = \frac{dy}{dx} = y'.$$

Furthermore, we write $f'' = \frac{d}{dx}\frac{d}{dx}f$ as $\frac{d^2}{dx^2}f$; the ith derivative is $\frac{d^i}{dx^i}f$. Of course, if f's variable were not x but rather t, then we would write $\frac{d}{dt}f$ instead of $\frac{d}{dx}f$.

Here are some common derivatives. If $f(x) = k$ is constant, then $f'(x) = 0$. The derivative of x^k is kx^{k-1} (this is called the *power rule*). On the other hand, the derivative of k^x is $k^x \cdot \ln k$; in particular, the derivative of e^x is e^x. The derivative of $\ln x$ is $1/x$. The derivative of $\sin x$ is $\cos x$, and the derivative of $\cos x$ is $-\sin x$. To compute the derivative of a more complicated function, we break the function down:

Theorem. Let k be a constant, and let f and g be two differentiable functions. Then
A. $(f + g)' = f' + g'$,
B. $(k \cdot g)' = k \cdot g'$,
C. $(f \cdot g)' = f \cdot g' + g \cdot f'$ (note that when f is constant this reduces to part B), and
D. $(f(g))' = f'(g) \cdot g'$ (this is called the *chain rule*).

For example, the derivative of $3x^2e^x$ is $\frac{d}{dx}(3x^2e^x) = 3 \cdot \frac{d}{dx}(x^2e^x) = 3(x^2 \cdot \frac{d}{dx}e^x + e^x \cdot \frac{d}{dx}x^2) = 3(x^2 \cdot e^x + e^x \cdot 2x) = 3e^x(x^2 + 2x)$. Here we have used part B, part C, the power rule, and $\frac{d}{dx}e^x = e^x$. The derivative of $\sin(3x^2e^x)$ is then $\cos(3x^2e^x) \cdot \frac{d}{dx}(3x^2e^x) = \cos(3x^2e^x) \cdot 3e^x(x^2 + 2x)$, by the chain rule.

For another example, suppose x and y are implicitly related by $x^2 + y^2 = 1$. Then, differentiating this equation with respect to x, we obtain $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$, so $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$, so $2x + 2y\frac{d}{dx}(y) = 0$. (In this last step we have used the chain rule.) Solving this equation for $\frac{d}{dx}(y)$, we find that $\frac{dy}{dx} = -x/y$ (if $y \neq 0$). Since $x^2 + y^2 = 1$ describes the unit circle, we see that the line tangent to the unit circle at (x, y), where $y \neq 0$, has slope $-x/y$. (Where $y = 0$ the tangent is vertical.)

If $f(x)$ has an inverse function $g(y)$, then $g'(y) = 1/f'(x)$ wherever $f'(x)$ and $g'(y)$ are nonzero.

It is a theorem that f is continuous wherever it is differentiable. However, a function can be continuous without being differentiable. For example, the functions $x^{2/3}$ and $|x|$ are not differentiable at 0, although they are continuous there (and differentiable everywhere else).