MATH 206, SPRING 2007, HOMEWORK 4

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The first three questions arise from our discussion of the action principle and the Euler-Lagrange equations. There is a concise treatment of that stuff posted to the class web site now, if you want to review it.

A. Suppose that a ray or particle of light in $\mathbb{R}^2$ starts out from $(0, 1)$, travels in a straight line, bounces off the x-axis (which is a mirror) at some point $(x, 0)$, travels on in a straight line, and ends up at the point (a, b) in the first quadrant. So its path consists of two line segments, which it traverses at a constant speed c. Fermat's principle of least time (a precursor to the action principle) says that the light travels along the path that takes the shortest time.

1. Prove that the derivative (with respect to x) of travel time is zero when $x = \frac{a}{1+b}$. (In fact, this x minimizes travel time. You should also find the answer $x = \frac{a}{1-b}$. To convince yourself that $x = \frac{a}{1+b}$ is a minimum and $x = \frac{a}{1-b}$ is not, pick a few values for a and b and graph the travel time. A rigorous proof is more algebra than it's worth.)

2. The two line segments make two acute angles with the x-axis. Prove that these angles are equal when $x = \frac{a}{1+b}$. You have just proved that when light reflects, its *angle of incidence* and its *angle of reflection* are equal.

B. Let $\vec{\gamma} : [t_0, t_1] \to \mathbb{R}^3$ be a smooth curve such that

$$\int_{t_0}^{t_1} \vec{\gamma}(t) \cdot \vec{\beta}(t)\, dt = 0$$

for all smooth vector fields $\vec{\beta} : [t_0, t_1] \to \mathbb{R}^3$ with $\vec{\beta}(t_0) = \vec{\beta}(t_1) = \vec{0}$. Prove that $\vec{\gamma}$ must be identically $\vec{0}$.

(This is the claim I use at the end of the proof that stationary points of the action satisfy the Euler-Lagrange equations. Hint: First set $\vec{\beta} = \vec{\gamma}$ and conclude that $\vec{\gamma} \equiv \vec{0}$. Unfortunately, this is not a proof, since $\vec{\gamma}$ is not a valid $\vec{\beta}$, since we have no reason to believe that $\vec{\gamma}(t_0) = \vec{\gamma}(t_1) = \vec{0}$. Try to work around that problem.)

C. Instead of studying a single curve $\vec{\alpha}(t)$, consider now a smooth one-parameter family of curves

$$\vec{\alpha}_\epsilon(t) = \vec{\alpha}(\epsilon, t)$$

for $\epsilon \in \mathbb{R}$. This is like a variation, except that the endpoints are not necessarily fixed. Assume that all $\vec{\alpha}_\epsilon$ satisfy the Euler-Lagrange equations. For simplicity, assume also that ϵ and t are independent, so that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \epsilon}\vec{\alpha} = \frac{\partial}{\partial \epsilon}\frac{\partial}{\partial t}\vec{\alpha}.$$

Now here's the important part: Suppose that our Lagrangian $L(t, \vec{x}, \vec{v})$ is invariant under changes in ϵ, meaning that

$$\frac{d}{d\epsilon}L(t, \vec{\alpha}_\epsilon, d\vec{\alpha}_\epsilon/dt) \equiv 0.$$

Let

$$C = \frac{\partial L}{\partial \vec{v}} \cdot \frac{\partial \vec{\alpha}}{\partial \epsilon}.$$

Prove that $dC/dt \equiv 0$, so that C is constant over time.

(This innocuous-looking result is surprisingly important in modern physics. It says that any one-parameter "symmetry" of L automatically yields a conserved quantity C.)

D. In class we've solved Section 2.3 #15B, which says that arc length is independent of parametrization. Using similar methods — i.e. the chain rule — prove that curvature is independent of parametrization. (Your proof must be direct. You may not just say "arc length is independent, so everything defined from it is too" — although that's true.)

E. Section 2.2 #2.

F. Section 2.2 #4.

You no longer need to solve Section 2.2 #7 on this homework; it has been moved to the next assignment.