# MATH 206, SPRING 2007, HOMEWORK 5

REVISED 13 FEBRUARY 2007

A. Section 2.2 Problem 7.

B. The torus $T \subseteq \mathbb{R}^3$ is proved to be a regular surface in Section 2.2 Example 4, but no parametrizations are given. Find an explicit parametrization $\vec{x} : (-\pi, \pi) \times (-\pi, \pi) \to T$ that covers as much of $T$ as possible. (Hint: Try cylindrical coordinates.) Adapt this parametrization into two more, $\vec{y}$ and $\vec{z}$, such that the three parametrizations together cover $T$. Draw three pictures to illustrate the three parametrizations, respectively — especially noting the points that they miss.

The rest of the questions concern the following construction. Let $S$ be the unit sphere

$$S = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subseteq \mathbb{R}^3$$

and $\vec{p} = (0, 0, 1)$ its "north pole". For any point $\vec{a} = (a_1, a_2, 0)$ in the $x_1$-$x_2$-plane, there is a unique line $L$ through $\vec{a}$ and $\vec{p}$, and this line intersects $S$ in exactly two points. One is $\vec{p}$; call the other one $\vec{b}$.

Define $\vec{x} : \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ by sending $(u, v)$ to $(u, v, 0)$ and then sending $(u, v, 0)$ to its corresponding $\vec{b}$-point on $S$. It's easy to see that this map $\vec{x}$ is injective. The inverse map $\vec{x}^{-1} : S \to \mathbb{R}^2$ that sends $\vec{b}$ to $\vec{a}$ (and then forgets about $a_3 = 0$) is called *stereographic projection*.

This projection finds use in several different areas of mathematics and science. We will study it repeatedly this semester. The treatment here is slightly different from that of Section 2.2 #16.

C. Where does $\vec{x}$ send $(0, 0)$? Where does it send the $u$-axis $\{v = 0\}$, the $v$-axis $\{u = 0\}$, and the unit circle $\{u^2 + v^2 = 1\}$? Where does it send the open unit disk $\{u^2+v^2 < 1\}$ and the open set $\{u^2+v^2 > 1\}$? Does the image of $\vec{x}$ completely cover $S$? If not, which points are left out? (At these points stereographic projection is not defined, of course.) Answer these questions and draw a detailed picture that illustrates all of them. Rigorous proofs are not required; you're just trying to explain the parametrization/projection to a classmate who doesn't understand it.

D. Prove the formula

$$\vec{x}(u, v) = \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

E. Prove that $x_3(u, v) \to 1$ as $|(u, v)| \to \infty$. Explain the statement *The sphere is the plane with a single "point at infinity" added.*

F. Compute the differential

$$d\vec{x}_{(u,v)} : T_{(u,v)}\mathbb{R}^2 \to T_{\vec{x}(u,v)}S$$

(as a $3 \times 2$ matrix, with respect to the standard bases). Simplify.

G. Prove that $d\vec{x}_{(u,v)}$ is injective. (Since $\vec{x}$ is injective and $C^\infty$, we now know that $\vec{x}$ is a *bona fide* parametrization.)

H. Explain (in pictures and words rather than formulas) how the sphere can be covered completely by $\vec{x}$ and one other, similar parametrization $\vec{y}$.