MECHANICS, QUICKLY

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Abstract. This brief paper describes the action principle, the Euler-Lagrange equations, Newton's second law of motion, and Noether's theorem. It is suitable for students who have mastered multivariable calculus.

1. Introduction

Around 1662 Pierre de Fermat formulated a *principle of least time*, which says that light travels from one point to another by the path that takes the least time. In the simplest scenarios the principle implies that the path is a straight line. Among paths that bounce once off a mirror, the principle implies that the angles of incidence and reflection are equal (see Exercise 1.1). Similarly, when light passes from one medium into another at a non-right angle to their interface, and the speeds of light in the two media are unequal, then the light bends at the interface according to Snell's law of refraction; this law is easily reproduced by minimizing the travel time of the light from a point in the first medium to a point in the second.

Over the centuries the principle of least time has been generalized repeatedly. First, the travel time was replaced by a another quantity, the *action*; it is the integral of a function — the *Lagrangian* — that describes the physical problem under consideration. Fermat's principle became the *principle of least action*, which states that physical trajectories correspond to minima of the action. Later it was realized that physical phenomena arise not just from minima of the action, but from all *stationary points* — trajectories where the derivative of the action vanishes. This is the content of the modern *action principle*.

In classical mechanics the action principle is equivalent to a system of differential equations called the *Euler-Lagrange equations*, which is in turn equivalent to Newton's second law of motion, $\vec{F} = m\vec{a}$. However, many physical problems are easier to study using Lagrangians than they are using Newton's second law. For example, any one-parameter *symmetry* of the Lagrangian automatically produces a conservation law (Noether's theorem).

The Lagrangian approach is so powerful that it is even used to describe nonclassical physics such as quantum mechanics. In this context, the action principle seems to arise from an even more fundamental principle, which expresses the probability amplitudes of quantum mechanics as integrals over the space of all possible particle paths. Although this *path integral* idea was first proposed by Richard Feynman around 1948, the integrals are difficult to define rigorously, and they are still not fully understood.

This paper describes how the action principle leads to the Euler-Lagrange equations, Newton's second law of motion. It also discusses Noether's theorem.

Exercise 1.1. *Suppose that a ray or particle of light in* $\mathbb{R}^2$ *starts out from* $(0, 1)$*, travels in a straight line, bounces off the* x*-axis (which is a mirror) at some point*

$(x, 0)$, travels on in a straight line, and ends up at the point (a, b) in the first quadrant. So its path consists of two line segments, which make two acute angles with the x-axis, called the angle of incidence *and the* angle of reflection. *Assuming that the speed of light is some constant c, express the travel time as a function of x. Prove that it's minimized at $x = \frac{a}{1+b}$, and that the angles of incidence and reflection are equal there.*

2. Action Principle

We wish to study physics in $\mathbb{R}^n$. If you like, you can set $n = 2$ or $n = 3$. It is helpful to consider the *phase space* or *configuration space* $X = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ consisting of all times t, locations $\vec{x}$, and velocities $\vec{v}$. Any smooth curve

$$\vec{\alpha} : [t_0, t_1] \to \mathbb{R}^n,$$

perhaps representing the trajectory of a point particle, induces a smooth curve

$$t \mapsto (t, \vec{\alpha}(t), \vec{\alpha}'(t))$$

in the phase space. Physical law is imposed on the particle in the form of a smooth function

$$L(t, \vec{x}, \vec{v}) : X \to \mathbb{R},$$

called the *Lagrangian*. (An example will be be given later.) Define the *action* to be the integral

$$I(\vec{\alpha}) = \int_{t_0}^{t_1} L(t, \vec{\alpha}, \vec{\alpha}')\, dt.$$

The action principle says that curves $\vec{\alpha}$ at which "$dI/d\vec{\alpha} = 0$" are the curves that actually arise in the physical system governed by L.

But what does it mean to take the derivative $dI/d\vec{\alpha}$ of I with respect to curves $\vec{\alpha}$? The idea belongs to that subject of math called the calculus of variations. A *variation* of $\vec{\alpha} : [t_0, t_1] \to \mathbb{R}^n$ is a one-parameter family of curves of the form

$$\vec{\alpha} + \epsilon\vec{\beta},$$

where $\vec{\beta} : [t_0, t_1] \to \mathbb{R}^n$ is a smooth vector field on the interval $[t_0, t_1]$ satisfying

$$\vec{\beta}(t_0) = \vec{\beta}(t_1) = \vec{0}.$$

We imagine the vectors $\vec{\beta}(t)$ scaled by the (small, time-independent) scalar ϵ and attached to the points $\vec{\alpha}(t)$, with the effect of perturbing the curve $\vec{\alpha}$ slightly while keeping its endpoints $\vec{\alpha}(t_0)$ and $\vec{\alpha}(t_1)$ fixed.

To say that $dI/d\vec{\alpha} = 0$ is to say that I is infinitesimally unchanged by any variation in $\vec{\alpha}$ — that is,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(\vec{\alpha} + \epsilon\vec{\beta}) = 0$$

for all $\vec{\beta}$. Such $\vec{\alpha}$ are called *stationary points* of the action. The action principle says that physical behavior arises at stationary points.

3. Euler-Lagrange Equations

In this section and the next we illustrate the action principle as it occurs in classical mechanics. Let

$$\begin{aligned}\frac{\partial L}{\partial \vec{x}} &= (\partial L/\partial x_1, \ldots, \partial L/\partial x_n), \\ \frac{\partial L}{\partial \vec{v}} &= (\partial L/\partial v_1, \ldots, \partial L/\partial v_n)\end{aligned}$$

denote the gradients of $L(t, \vec{x}, \vec{v})$ with respect to $\vec{x}$ and $\vec{v}$. In this notation, the *Euler-Lagrange equations* are

$$\frac{\partial L}{\partial \vec{x}} = \frac{\partial}{\partial t}\frac{\partial L}{\partial \vec{v}}.$$

Theorem 3.1. *If $\vec{\alpha}$ is a stationary point of the action, then it satisfies the Euler-Lagrange equations:*

$$\frac{\partial L}{\partial \vec{x}}(t, \vec{\alpha}, \vec{\alpha}') = \frac{\partial}{\partial t}\frac{\partial L}{\partial \vec{v}}(t, \vec{\alpha}, \vec{\alpha}').$$

Proof. Assume that $\vec{\alpha}$ is a stationary point. Then for all $\vec{\beta}$,

$$\begin{aligned}0 &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I(\vec{\alpha} + \epsilon\vec{\beta}) \\ &= \int_{t_0}^{t_1} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(t, \vec{\alpha} + \epsilon\vec{\beta}, \vec{\alpha}' + \epsilon\vec{\beta}')\, dt \\ &= \int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \frac{\partial L}{\partial x_i}(t, \vec{\alpha} + \epsilon\vec{\beta}, \vec{\alpha}' + \epsilon\vec{\beta}')\beta_i + \sum_{i=1}^{n} \frac{\partial L}{\partial v_i}(t, \vec{\alpha} + \epsilon\vec{\beta}, \vec{\alpha}' + \epsilon\vec{\beta}')\beta_i'\right]_{\epsilon=0} dt \\ &= \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} \frac{\partial L}{\partial x_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i + \sum_{i=1}^{n} \frac{\partial L}{\partial v_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i'\right) dt.\end{aligned}$$

Now integration by parts tells us that

$$\begin{aligned}\int_{t_0}^{t_1} \frac{\partial L}{\partial v_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i'\, dt &= \left[\frac{\partial L}{\partial v_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{\partial}{\partial t}\frac{\partial L}{\partial v_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i\, dt \\ &= -\int_{t_0}^{t_1} \frac{\partial}{\partial t}\frac{\partial L}{\partial v_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i\, dt.\end{aligned}$$

(The boundary term vanishes since $\beta_i(t_0) = \beta_i(t_1) = 0$.) Therefore

$$\begin{aligned}0 &= \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} \frac{\partial L}{\partial x_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i - \frac{\partial}{\partial t}\frac{\partial L}{\partial v_i}(t, \vec{\alpha}, \vec{\alpha}')\beta_i\right) dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \vec{x}}(t, \vec{\alpha}, \vec{\alpha}') - \frac{\partial}{\partial t}\frac{\partial L}{\partial \vec{v}}(t, \vec{\alpha}, \vec{\alpha}')\right) \cdot \vec{\beta}\, dt.\end{aligned}$$

By Exercise 3.2 below,

$$\frac{\partial L}{\partial \vec{x}}(t, \vec{\alpha}, \vec{\alpha}') - \frac{\partial}{\partial t}\frac{\partial L}{\partial \vec{v}}(t, \vec{\alpha}, \vec{\alpha}') = \vec{0},$$

as desired. □

Exercise 3.2. *Let* $\vec{\gamma} : [t_0, t_1] \to \mathbb{R}^n$ *be a smooth curve such that*

$$\int_{t_0}^{t_1} \vec{\gamma}(t) \cdot \vec{\beta}(t)\, dt = 0$$

for all smooth vector fields $\vec{\beta} : [t_0, t_1] \to \mathbb{R}^n$ *with* $\vec{\beta}(t_0) = \vec{\beta}(t_1) = \vec{0}$. *Prove that* $\vec{\gamma}$ *must be identically* $\vec{0}$.

4. Newton's Second Law of Motion

In this section we describe the classical mechanics of a point particle of mass m under the influence of a conservative force field. The Lagrangian is the difference between the *kinetic energy* $K = \frac{1}{2}m|\vec{v}|^2$ and the *potential energy* $P = P(\vec{x})$. Kinetic energy is the energy of the particle's motion, while potential energy is extra energy possessed by the particle by virtue of its position $\vec{x}$ in a force field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$. It is the work required to move the particle from some predetermined reference point $\vec{0}$ to $\vec{x}$ along any curve $\vec{\gamma}$. (The assumption that the work is independent of $\vec{\gamma}$ is what makes the force *conservative*.) If $\vec{\gamma} = \vec{\gamma}(s)$ is parametrized by arc length s with $\vec{\gamma}(0) = \vec{0}$, then the work done by $\vec{F}$ from $s = 0$ to $s = t$ is defined as

$$P(\vec{\gamma}(t)) = \int_0^t -\vec{F}(\vec{\gamma}(s)) \cdot \vec{\gamma}'(s)\, ds.$$

Exercise 4.1. *Prove that force is the negative gradient of the potential:*

$$\vec{F}(\vec{x}) = -\frac{\partial P}{\partial \vec{x}}.$$

Exercise 4.2. *Let* $P(\vec{x}) = -1/|\vec{x}|$. *Prove that*

$$-\frac{\partial P}{\partial \vec{x}} = -\frac{1}{|\vec{x}|^2}\frac{\vec{x}}{|\vec{x}|}.$$

This is a force of magnitude $1/|\vec{x}|^2$ *directed toward the origin, like the forces in Newton's law of gravitation or Coulomb's law of electrical attraction.*

If $L(t, \vec{x}, \vec{v}) = K - P = \frac{1}{2}m|\vec{v}|^2 - P(\vec{x})$, then Exercise 4.1 tells us that

$$\begin{aligned} \frac{\partial L}{\partial \vec{x}} &= -\frac{\partial}{\partial \vec{x}}P = \vec{F}, \\ \frac{\partial}{\partial t}\frac{\partial L}{\partial \vec{v}} &= \frac{\partial}{\partial t}\frac{\partial}{\partial \vec{v}}\frac{1}{2}m\vec{v} \cdot \vec{v} = \frac{d}{dt}m\vec{v} = m\vec{a}, \end{aligned}$$

where $\vec{a}$ is acceleration. Thus the Euler-Lagrange equations reduce to Newton's second law of motion, $\vec{F} = m\vec{a}$.

5. Noether's Theorem

Consider now a smooth one-parameter family of curves

$$\vec{\alpha}_\epsilon(t) = \vec{\alpha}(\epsilon, t)$$

for $\epsilon \in \mathbb{R}$. This is like a variation, except that the endpoints are not necessarily fixed. Suppose that all $\vec{\alpha}_\epsilon$ satisfy the Euler-Lagrange equations, and that ϵ and t are independent:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \epsilon}\vec{\alpha} = \frac{\partial}{\partial \epsilon}\frac{\partial}{\partial t}\vec{\alpha}.$$

Theorem 5.1. *Assume the preceding hypotheses and that the Lagrangian* $L(t, \vec{x}, \vec{v})$ *is invariant under changes in* ϵ*, meaning that*

$$\frac{d}{d\epsilon} L(t, \vec{\alpha}_\epsilon, d\vec{\alpha}_\epsilon/dt) \equiv 0.$$

Let

$$C = \frac{\partial L}{\partial \vec{v}} \cdot \frac{\partial \vec{\alpha}}{\partial \epsilon}.$$

Then $\partial C/\partial t \equiv 0$.

Exercise 5.2. *Prove the theorem.*

This innocuous-looking result is surprisingly important in modern physics. It says that any one-parameter *symmetry* of a Lagrangian L automatically yields a *conserved quantity* C. It was first proved by Emmy Noether around 1915.