Math 206-01, Spring 2007, Exam 1

1. Instructions

This test begins for you when you open this packet. It ends at 4:20 PM on Tuesday 20 February 2007, although you are welcome to hand in your work earlier. There is no time limit; I expect it to take at least as long as a homework assignment. It is a good idea to get started early and work on it over multiple days. If you have any questions about the exam or its rules, then ask me for clarification. Here are the rules:

- You may freely consult all of this class' material: the do Carmo textbook, your class notes, your old homework, and the materials on the class web site. If you missed a lecture and need to copy someone else's class notes, do so before either of you begins the exam.
- You may assume all results that we have proved in class and in the assigned homework. You do not have to prove or reprove them on this test. On the other hand, you may not cite results that we have not studied.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may not use a calculator or computer, except to view the class web site materials or to type up your answers.
- You may talk to me in private or over e-mail. I will try to check my e-mail several times per day. You may ask clarifying questions for free. If you're really stuck on a problem, then you may ask for a hint. A hints costs you some points, based on how valuable I judge it to be. For this reason, I will not give you a hint unless you explicitly and unambiguously request it. The opportunity to ask questions is another reason to get started early.
- You may not discuss the exam in any way (spoken, written, pantomime, semaphore, etc.) with anyone but me until the due date, 4:20 PM on Tuesday 20 February 2007, even if you finish earlier. Before then you will inevitably see your classmates around campus. You must refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.

There are four problems, with two, four, six, and four parts, respectively. Some of the parts are quite small, existing only to expedite grading. In some problems the later parts require solution of the earlier parts; in other problems, they do not; in still other problems, they are simply unrelated.

Your solutions should be polished (concise, neat, and well-written, employing complete sentences with punctuation) and self-explanatory. Always show enough work so that a classmate could follow your solutions. Do not show anything unnecessary (scratch work, false starts, circuitous reasoning, etc.). Answers should always be exact and simplified. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Submit your solutions in a single stapled packet, presented in the order they were assigned. Write and sign the honor pledge on the packet.

Partial credit is often awarded. Exam grades will be "curved" — by this I mean that there are no predetermined scores required for grades A, B, C, D, F. Good luck.

2. Bézier Splines

Let $\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^3$ be any four points. Define a parametrized curve $\vec{\alpha} : [0,1] \to \mathbb{R}^3$ by the formula

$$\vec{\alpha}(t) = (1-t)^3 \vec{a}_0 + 3t(1-t)^2 \vec{a}_1 + 3t^2(1-t)\vec{a}_2 + t^3 \vec{a}_3.$$

This is the cubic *Bézier curve* for the *control points* $\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3$. Suppose that $\vec{b}_0, \vec{b}_1, \vec{b}_2, \vec{b}_3$ is another sequence of control points, with Bézier curve $\vec{\beta}$. Under suitable conditions, $\vec{\alpha}$ and $\vec{\beta}$ can be joined end-to-end to construct a more complicated curve, which is then called a *spline*. Such splines are used heavily in computer graphics, such as for drawing the letters that you are reading right now. There is a similar notion of Bézier surfaces, that we may study later in the semester.

A. Compute the end points $\vec{\alpha}(0)$ and $\vec{\alpha}(1)$. Compute the tangent vectors $\vec{\alpha}'(0)$ and $\vec{\alpha}'(1)$. Using these results, draw a picture that illustrates how the points $\vec{a}_0, \vec{a}_1, \vec{a}_2, \vec{a}_3$ "control" $\vec{\alpha}$.

B. What conditions must the $\vec{a}_i$ and $\vec{b}_i$ meet, if $\vec{\alpha}(1)$ is to coincide with $\vec{\beta}(0)$? What conditions must they meet, if the two curves are to have the same tangent vector there?

3. Zero Curvature

Usually we prefer curves with nonvanishing curvature over those whose curvature vanishes somewhere. In this problem we explore that bias. We seek a curve $\vec{\alpha}(s)$, parametrized by arc length, with constant curvature $k(s) \equiv 0$ and constant torsion $\tau(s) \equiv 1$, that satisfies the Frenet equations with initial conditions $\vec{t}(0) = (0,0,1)$, $\vec{n}(0) = (1,0,0)$, and $\vec{b}(0) = (0,1,0)$.

A. Rewrite and simplify the Frenet equations using these assumptions.

B. Find one orthonormal solution $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ to the differential equations from Part A and the initial conditions.

C. What parametrized curve $\vec{\alpha}(s)$ arises from the solution to Part B? Give a formula. Also sketch the curve, indicating how the frame $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ moves along the curve.

D. How would the curve $\vec{\alpha}(s)$ have been different if I had specified $\tau(s) \equiv 0$ instead of $\tau(s) \equiv 1$? What is the role of torsion in this problem? (Probably at least a few sentences are required to answer these questions.)

4. Riparian Erosion

This problem uses parametrized curves to model erosion along a river. No knowledge of fluid dynamics or sediment transport is necessary. The model may or may not be scientifically accurate; compromises have been made to keep it tractable. Do not assume that the curves in this problem are parametrized by arc length.

A. As a prelude, prove that for any plane curve $\vec{\alpha}(t) = (x(t), y(t)) : \mathbb{R} \to \mathbb{R}^2$,

$$-\left|\vec{\alpha}'\right|^4 k\vec{n} = \left(y'(x'y'' - y'x''), -x'(x'y'' - y'x'')\right).$$

Viewed from a high altitude, a river flowing across a flat landscape looks a lot like a plane curve $\vec{\alpha}(t)$. That is, $\vec{\alpha}(t) = (x(t), y(t))$ is the position at time t of some molecule of water flowing along the river, and the other molecules pass through the same points at other times t. For example, a meandering river might be described by $\vec{\alpha}(t) = (t, \sin t)$, while a straight, fast river might be described by $\vec{\alpha}(t) = (10t, 0)$.

However, a river continually erodes its banks, so the course of the river changes gradually over time. So the river is really a one-parameter family of curves

$$\vec{\alpha}_s(t) = \vec{\alpha}(t, s) = (x(t, s), y(t, s)).$$

Although both t and s represent time, they have very different scales — seconds vs. years, perhaps — and we should view them as independent variables.

Here's how erosion works. At any bend in the river, the water tends to deposit sediment on the inside of the bend and remove sediment from the outside of the bend, for reasons that lie beyond the scope of this problem. Therefore the bend in the river moves; over time it comes to bend even further outward than it used to. This is why rivers develop meanders. In contrast, straight portions of the river do not change much as erosion progresses. One more thing: The rate of erosion increases dramatically as the speed of the water increases.

B. Based on the description of erosion just given, explain why $\vec{\alpha}(t, s) = (x(t, s), y(t, s))$ might satisfy the following differential equations. The prime $'$ now denotes $\partial/\partial t$, rather than d/dt.

$$\begin{aligned} \frac{\partial x}{\partial s} &= y'(x'y'' - y'x''), \\ \frac{\partial y}{\partial s} &= -x'(x'y'' - y'x''). \end{aligned}$$

Assuming that the initial course of the river is $\vec{\alpha}(t, 0) = (t, t^2)$, we will now solve the differential equations using power series. (If you have never done such stuff before, give it a try, and ask questions if necessary.) Suppose that $x(t, s)$ and $y(t, s)$ expand asymptotically as

$$\begin{aligned} x(t, s) &= \sum_{d=0}^{\infty} \sum_{i+j=d} a_{ij} t^i s^j &= a_{00} + a_{10}t + a_{01}s + a_{20}t^2 + a_{11}ts + a_{02}s^2 + \cdots, \\ y(t, s) &= \sum_{d=0}^{\infty} \sum_{i+j=d} b_{ij} t^i s^j &= b_{00} + b_{10}t + b_{01}s + b_{20}t^2 + b_{11}ts + b_{02}s^2 + \cdots. \end{aligned}$$

Finding x and y boils down to finding the constants a_{ij} and b_{ij}. To keep things tractable, forget about the $\cdots$, thereby truncating the series after the quadratic terms. In other words, let's just approximate x and y by their degree-2 Taylor polynomials.

C. What does the initial condition $\vec{\alpha}(t, 0) = (t, t^2)$ tell you about the coefficients a_{ij} and b_{ij} in the Taylor polynomials? Use this information to rewrite the differential equations from Part B explicitly.

D. Solve for $y(t, s)$ and $x(t, s)$. Cubic terms may arise; ignoring them does not make your life easier, so do not ignore them. Your solutions will have one undetermined parameter, such as a_{11}.

E. Set the undetermined parameter equal to 1. (This is another initial condition, but not one worth explaining.) Give a simplified, exact formula for the curve of the river at time $s = 0.1$.

F. In a single graph, sketch the river at times $s = 0$ and $s = 0.1$, based on the solution to Part E. Your sketch should be precise enough that the x- and y-intercepts are approximately correct.

5. Stereographic Projection

Recall from our recent homework the stereographic projection and its inverse $\vec{x} : \mathbb{R}^2 \to S$ given by

$$\vec{x}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

A. How does the differential $d\vec{x}$ affect norms? To answer this, let $(u, v) \in \mathbb{R}^2$ and $\vec{q} \in T_{(u,v)}\mathbb{R}^2 \cong \mathbb{R}^2$, and derive an explicit, simplified formula for $\left| d\vec{x}_{(u,v)}\vec{q} \right|$ in terms of u, v, $\vec{q}$, $|\vec{q}|$, etc.

B. At which points (u, v) does the differential preserve norms?

C. Prove that the differential $d\vec{x}$ preserves angles. That is, for all $(u, v) \in \mathbb{R}^2$ and all $\vec{q}, \vec{r} \in T_{(u,v)}\mathbb{R}^2$, the angle between $d\vec{x}_{(u,v)}\vec{q}$ and $d\vec{x}_{(u,v)}\vec{r}$ is equal to the angle between $\vec{q}$ and $\vec{r}$.

D. In $\mathbb{R}^2$, let C be the circle of radius 1 centered at $(1, 0)$. Prove that the image of C under $\vec{x}$ is a circle in $S \subseteq \mathbb{R}^3$. To the best of your ability, draw this circle on the sphere. (In fact, $\vec{x}$ sends all circles and lines in $\mathbb{R}^2$ to circles in S! Prove that if you like, but it doesn't earn any more points.)