Math 206-01, Spring 2007, Exam 2

The rules of this exam are identical to those of our previous exam, except that you are now allowed to use Mathematica. For your convenience I have repeated the rules, with this modification, below.

This test begins for you when you open this packet. It ends at 4:20 PM on Tuesday 10 April 2007, although you are welcome to hand in your work earlier. There is no time limit. It is a good idea to get started early and work on it over multiple days. If you have any questions about the exam or its rules, then ask me for clarification.

- You may freely consult all of this class' material: the do Carmo textbook, your class notes, your old homework, and the materials on the class web site. If you missed a lecture and need to copy someone else's class notes, do so before either of you begins the exam.
- You may assume all results that we have proved in class and in the assigned homework. You do not have to prove or reprove them on this test. On the other hand, you may not cite results that we have not studied.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may use a computer for these four purposes: viewing the class web site materials, running Mathematica, typing up your answers, and e-mailing questions to me. You may not use a computer or calculator for any other purpose.
- You may talk to me in private or over e-mail. I will try to check my e-mail several times per day. You may ask clarifying questions for free. If you're really stuck on a problem, then you may ask for a hint. A hints costs you some points, based on how valuable I judge it to be. For this reason, I will not give you a hint unless you explicitly and unambiguously request it. The opportunity to ask questions is another reason to get started early.
- You may not discuss the exam in any way (spoken, written, pantomime, semaphore, etc.) with anyone but me until the due date, even if you finish earlier. During the exam you will inevitably see your classmates around campus. You must refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.

Your solutions should be polished (concise, neat, and well-written, employing complete sentences with punctuation) and self-explanatory. Always show enough work so that a classmate could follow your solutions. When using Mathematica, show all necessary functions and computations, so that I can understand and verify your solution. Do not show anything unnecessary (scratch work, false starts, circuitous reasoning, etc.). Answers should always be exact and simplified. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Submit your solutions in a single stapled packet, presented in the order they were assigned. Write and sign the honor pledge on the packet.

Partial credit is often awarded. Exam grades will be "curved" — by this I mean that there are no predetermined scores required for grades A, B, C, D, F. Good luck.

1. Mean Curvature Zero

Let $S \subseteq \mathbb{R}^3$ be a surface whose mean curvature is zero at every point. Assume furthermore that S has no planar points. For every point $p \in S$, prove that every neighborhood of p crosses the tangent plane T_pS — that is, there are points in the neighborhood on both sides of the tangent plane. (For this problem, you may cite without proof any result from any section of the book that we've studied so far, even if we did not prove that particular result in class.)

2. Hemispheres vs. Disks

Let S be the southern hemisphere (the $z < 0$ part) of the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Let D_R be the open disk of radius R in $\mathbb{R}^2$, centered at the origin. I'd like to find an area-preserving diffeomorphism between S and D_R, for some R.

A. Prove that no diffeomorphism between S and D_R can be both conformal and area-preserving.

B. Does a (nonconformal) area-preserving diffeomorphism exist for some R? If so, then give a formula (you get to pick the R) and explain it; if not, then explain why (for all R).

3. Covariant Derivative

A. In Mathematica, write a function of the form

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covar[x_, alpha_, w_] := ...;
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to compute the covariant derivative, in the surface parametrized locally by $\vec{x}$, of the vector field $\vec{w}$ along the curve $\vec{\alpha}$. Here $\vec{\alpha}$ is a parametric curve in the domain of $\vec{x}$, so the curve on the surface is actually $\vec{x} \circ \vec{\alpha}$. The vector field $\vec{w}$ is given as (w_1, w_2), meaning $w_1(t)\vec{x}_1 + w_2(t)\vec{x}_2$.

(If you cannot write the function in this form, then you will still earn full credit by simply listing, in a polished and easy-to-understand manner, the sequence of Mathematica commands needed to produce the covariant derivative for any given $\vec{x}$, $\vec{\alpha}$, $\vec{w}$.)

B. Let $\vec{\alpha}(t) = (ct, dt)$ for some constants c, d, not both zero. Let $\vec{w} = \vec{\alpha}'$ be the velocity vector of $\vec{\alpha}$. Let $\vec{x}$ be our usual stereographic parametrization. Compute the covariant derivative.

C. In the example of Part B, the trace of $\vec{x} \circ \vec{\alpha}$ is a geodesic, but $\vec{w} = \vec{\alpha}'$ is not parallel. Explain both of these statements. Explain why they do not contradict one another.

4. Apu de Beaumarchais

Consider $\mathbb{R}^2$ with standard coordinates (u^1, u^2). Let U be an open circular disk centered at the origin $(0, 0)$. Let $C^\infty U$ denote the set of differentiable functions $f : U \to \mathbb{R}$. Notice that it is a vector space under the operations of function addition and scalar multiplication. Define an *apu* to be a linear map $D : C^\infty U \to \mathbb{R}$ such that, for any functions $f, g \in C^\infty U$,

$$D(fg) = D(f) \cdot g(0, 0) + f(0, 0) \cdot D(g).$$

Notice that the apus also form a vector space. In this problem we prove a sequence of lemmas, some of which build on each other. If you cannot prove one of them, then assume it and try the later lemmas.

A. Prove that the two tangent vectors $\dfrac{\partial}{\partial u^1}$ and $\dfrac{\partial}{\partial u^2}$ at the origin are apus.

B. Let $f \in C^\infty U$ be a function that vanishes at the origin. For $i = 1, 2$, let

$$g_i(u^1, u^2) = \int_0^1 \left(\frac{\partial}{\partial u^i} f(u^1, u^2) \right) (tu^1, tu^2)\, dt.$$

Prove that

$$f(u^1, u^2) = \sum_{i=1}^{2} u^i g_i(u^1, u^2).$$

C. Again let f vanish at the origin and define g_i as above. Prove that

$$\left(\frac{\partial}{\partial u^i} f\right)(0,0) = g_i(0,0).$$

D. Again let f vanish at the origin, and let D be any apu. Prove that

$$D(f) = \sum_{i=1}^{2} (D(u^i)) \left(\frac{\partial}{\partial u^i} f\right)(0,0).$$

E. Extend the result of Part D to all $f \in C^\infty U$.

F. Prove that $\left\{\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right\}$ is a basis for the vector space of apus.

5. Exponential Map

The exponential map is described in Section 4.6 of our book. You'll want to read the first couple of pages there. This problem concerns the exponential map at the south pole of the unit sphere $\mathbb{S}^2$. Instead of working on the sphere itself, we will work on the plane $\mathbb{R}^2$ via stereographic projection. That is, let $p : \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ be the stereographic projection — the inverse of our usual stereographic parametrization $\vec{x}$.

A. Let $\vec{v}$ be a vector (of length less than π) tangent to the unit sphere at the south pole. Explain why $p(\exp_{(0,0,-1)} \vec{v}) = c\vec{v} \in \mathbb{R}^2$ for some scalar c.

B. Find c in terms of $\vec{v}$. If you cannot explicitly solve for c, then explain as clearly as possible its relation to $\vec{v}$.