Example 1

I'm thinking of a surface. It can be covered with two coordinate charts. The first chart has domain $U_1 = \mathbb{R}^2$ with coordinates u_1^1 and u_1^2, and the second chart has domain $U_2 = \mathbb{R}^2$ with coordinates u_2^1 and u_2^2. Let $U_{12} \subseteq U_1$ be everything except the origin:

$$U_{12} = \{(u_1^1, v_1^1) \neq (0,0)\}.$$

Similarly, let $U_{21} \subseteq U_2$ be everything except the origin. The transition maps between the two charts are $h_{12} : U_{21} \to U_{12}$ and $h_{21} : U_{12} \to U_{21}$ given by

$$\begin{aligned} (u_1^1, u_1^2) &= h_{12}(u_2^1, u_2^2) = \left(\frac{u_2^1}{(u_2^1)^2 + (u_2^2)^2}, \frac{-u_2^2}{(u_2^1)^2 + (u_2^2)^2} \right), \\ (u_2^1, u_2^2) &= h_{21}(u_1^1, u_1^2) = \left(\frac{u_1^1}{(u_1^1)^2 + (u_1^2)^2}, \frac{-u_1^2}{(u_1^1)^2 + (u_1^2)^2} \right). \end{aligned}$$

You can check that these two maps are inverses of each other and that both are smooth. In other words, they constitute a diffeomorphism that describes how points in the first chart match up with points in the second chart in the surface. Which surface am I thinking of?

Example 2

Let

$$U_1 = \{(u_1^1, u_1^2) : -\pi < u_1^1 < \pi, -\pi < u_1^2 < \pi\} \subseteq \mathbb{R}^2.$$

The second chart has domain

$$U_2 = \{(u_2^1, u_2^2) : 0 < u_2^1 < 2\pi, -\pi < u_2^2 < \pi\}.$$

Let $U_{12} \subseteq U_1$ be all points where $u_1^1 \neq 0$; notice that it has two disconnected pieces. Let $U_{21} \subseteq U_2$ be all points where $u_2^1 \neq \pi$, which is also disconnected. The transition maps are $h_{12} : U_{21} \to U_{12}$ and $h_{21} : U_{12} \to U_{21}$ given by

$$\begin{aligned} (u_1^1, u_1^2) &= h_{12}(u_2^1, u_2^2) = \begin{cases} (u_2^1, u_2^2) & \text{if } u_2^1 < \pi, \\ (u_2^1 - 2\pi, u_2^2) & \text{if } u_2^1 > \pi, \end{cases} \\ (u_2^1, u_2^2) &= h_{21}(u_1^1, u_1^2) = \begin{cases} (u_1^1, u_1^2) & \text{if } u_1^1 > 0, \\ (u_1^1 + 2\pi, u_1^2) & \text{if } u_1^1 < 0. \end{cases} \end{aligned}$$

You can check that these two maps are inverses of each other. Put simply, they identify the right half of U_1 with the left half of U_2 and *vice versa*. What surface be this?

Example 3

Repeat the same setup as in the preceding example, but now use the transition maps

$$\begin{aligned}(u_1^1, u_1^2) &= h_{12}(u_2^1, u_2^2) = \begin{cases} (u_2^1, u_2^2) & \text{if } u_2^1 < \pi, \\ (u_2^1 - 2\pi, -u_2^2) & \text{if } u_2^1 > \pi, \end{cases} \\ (u_2^1, u_2^2) &= h_{21}(u_1^1, u_1^2) = \begin{cases} (u_1^1, u_1^2) & \text{if } u_1^1 > 0, \\ (u_1^1 + 2\pi, -u_1^2) & \text{if } u_1^1 < 0. \end{cases}\end{aligned}$$

The only difference in this example is a little negation in the u^2-coordinates. The right half of U_1 is glued to the left half of U_2 as before, but the right half of U_2 is glued to the left half of U_1 with a flip. Was für eine Fläche ist es?

Definition

For each i in some set I (possibly infinite, even uncountable), let $U_i \subseteq \mathbb{R}^2$ be a nonempty open set. For each pair of distinct indices $j \neq i$ in I, let $U_{ij} \subseteq U_i$ be an open set, possibly empty, and let $h_{ij} : U_{ji} \to U_{ij}$ be a smooth map, such that $h_{ij}^{-1} = h_{ji}$. Altogether, the information

$$\{U_i\}_{i \in I}, \qquad \{U_{ij}\}_{j \neq i \in I}, \qquad \{h_{ij}\}_{j \neq i \in I}$$

constitutes what is called an *abstract surface*, or a *smooth two-dimensional manifold*. More generally, if we replace $\mathbb{R}^2$ with $\mathbb{R}^n$ in that definition, then we obtain the definition of a *smooth n-dimensional manifold*, for any integer $n \geq 0$.

Do Carmo's definition is slightly different. He begins with a set S (not assumed to be a subset of $\mathbb{R}^3$ — it could be any set at all) and then selects open sets U_i and functions $x_i : U_i \to S$ — any functions at all, since S has no special structure — such that

$$x_j^{-1} \circ x_i : x_i^{-1}(x_j(U_j)) \to x_j^{-1}(x_i(U_i))$$

is smooth.

The two definitions are equivalent. Starting from his definition, let $U_{ij} = x_i^{-1}(x_j(U_j)) \subseteq U_i$ and let $h_{ji} = x_j^{-1} \circ x_i$, and you get a surface according to my definition. Starting from my definition, let S be the disjoint union of all U_i, but with each point in U_{ij} identified with the corresponding point in U_{ji} (via h_{ij} and h_{ji}). Let x_i be the function that takes any point in U_i to its representative in S. Then you get do Carmo's definition.

My definition has the advantage that it doesn't posit the existence of S at the start; instead, S inevitably emerges from the definition. On the other hand, S is convenient for making subsequent definitions, so do Carmo is wise to feature it prominently.