STEREOGRAPHIC PROJECTION (IN PROGRESS)

JOSHUA R. DAVIS

Let S be the unit sphere

$$S = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subseteq \mathbb{R}^3$$

and $\vec{p} = (0, 0, 1)$ its "north pole". For any point $\vec{a} = (a_1, a_2, 0)$ in the x_1-x_2-plane, there is a unique line L through $\vec{a}$ and $\vec{p}$, and this line intersects S in exactly two points. One is $\vec{p}$; call the other one $\vec{b}$.

Define $\vec{x} : \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ by sending (u, v) to $(u, v, 0)$ and then sending $(u, v, 0)$ to its corresponding $\vec{b}$-point on S. It's easy to see that this map $\vec{x}$ is injective and hits every point on S except the north pole. Explicitly,

$$\vec{x}(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right).$$

Under $\vec{x}$, the unit circle is mapped to the equator and the unit disk is mapped to the southern hemisphere; the rest of the u-v-plane is mapped to the northern hemisphere. As $|(u, v)| = \sqrt{u^2 + v^2} \to \infty$, the third coordinate of $\vec{x}$ goes to 1, so $\vec{x}(u, v) \to \vec{p}$. Identifying the plane $\mathbb{R}^2$ with its image under $\vec{x}$, we see that the sphere is the plane with a single "point at infinity" added.

The inverse map $\vec{x}^{-1} : S \to \mathbb{R}^2$ that sends $\vec{b}$ to $\vec{a}$ (and then forgets about $a_3 = 0$) is called *stereographic projection* from $\vec{p}$. It is defined everywhere on S except at $\vec{p}$ itself.

One can define a parametrization around the north pole similarly, by sending (u, v) to $(u, -v, 0)$ and then inverting stereographic projection from the south pole. The result is a map $\vec{y} : \mathbb{R}^2 \to S$ given by

$$\vec{y}(s, t) = \left(\frac{2s}{1 + s^2 + t^2}, \frac{-2t}{1 + s^2 + t^2}, \frac{1 - s^2 - t^2}{1 + s^2 + t^2} \right).$$

It sends the unit circle to the equator and the unit disk to the northern hemisphere. It covers the sphere except for the south pole. Together, $\vec{x}$ and $\vec{y}$ cover the entire sphere. Their overlap is the sphere except for the two poles.

Notice the minus sign in the second coordinate of $\vec{y}$. This is necessary for the following beautiful magic to occur between $\vec{x}$ and $\vec{y}$. By simple algebra, one can check that

$$\begin{aligned} \vec{y}(s, t) &= \vec{x}\left(\frac{s}{s^2 + t^2}, \frac{-t}{s^2 + t^2} \right), \\ \vec{x}(u, v) &= \vec{y}\left(\frac{u}{u^2 + v^2}, \frac{-v}{u^2 + v^2} \right). \end{aligned}$$

These imply that the transition maps between the two charts are

$$\begin{aligned} (u, v) &= \left(\frac{s}{s^2 + t^2}, \frac{-t}{s^2 + t^2} \right), \\ (s, t) &= \left(\frac{u}{u^2 + v^2}, \frac{-v}{u^2 + v^2} \right). \end{aligned}$$

One can check that the Jacobians of the transition maps have positive determinant. So $\vec{x}$ and $\vec{y}$ induce the same orientation on their overlap in S. But their relationship is more special than just this.

Let's identify the real plane $\mathbb{R}^2$ with the complex line $\mathbb{C}$, so that $(u, v) \leftrightarrow u + iv \in \mathbb{C}$ and $(s, t) \leftrightarrow s + it \in \mathbb{C}$. Then the transition maps are

$$\begin{aligned} u + iv &= \frac{s - it}{s^2 + t^2} = \frac{1}{s + it}, \\ s + it &= \frac{u - iv}{u^2 + v^2} = \frac{1}{u + iv}. \end{aligned}$$

So the transition maps are just complex number inversion! Of course, the number $0 = 0 + i0 \leftrightarrow (0, 0)$ can't be inverted. The origin $(0, 0)$ in the s-t-plane maps to the north pole, which is the missing "point at infinity" in the u-v-plane. Thus the sphere "completes" the complex numbers $\mathbb{C}$ by adding one more element, called ∞, that makes statements such as "$1/0 = \infty$" rigorous. One can use this, for example, to construct an elegant theory of complex rational functions $p(z)/q(z)$, well-defined even where $q(z) = 0$.

Warning: The sphere of extended complex numbers $\mathbb{C} \cup \{\infty\}$ does not constitute a field, in the sense of abstract algebra. You cannot do arithmetic (particularly addition/subtraction) with ∞, even if it's very natural for geometry and complex analysis.