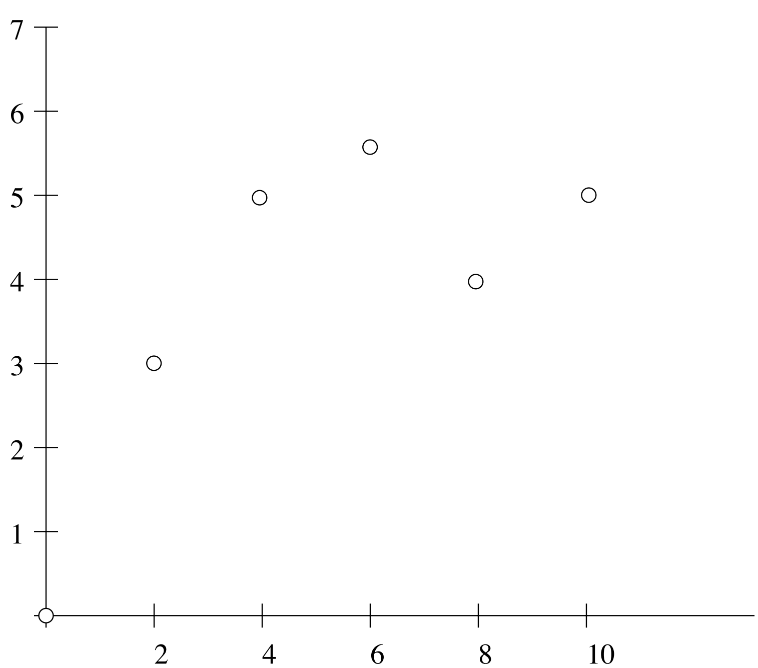
Math 31L-02 Spring 2007 Exam 1 Answers

1. Use the definition of the derivative to compute the derivative of the function $f(x) = \dfrac{1}{x+4}$.
Answer:

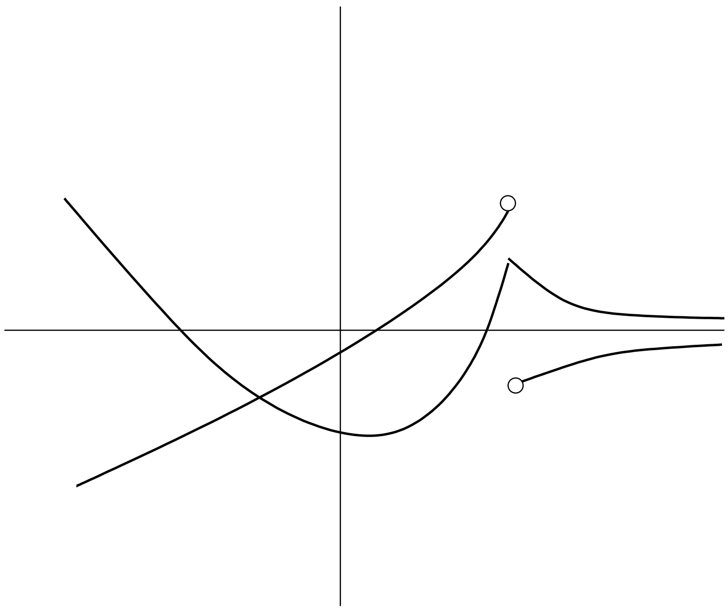
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h+4} + \frac{1}{x+4}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+4}{(x+h+4)(x+4)} - \frac{x+h+4}{(x+h+4)(x+4)}}{h} \\
&= \lim_{h\to 0} \frac{x+4-x-h-4}{h(x+h+4)(x+4)} \\
&= \lim_{h\to 0} \frac{-h}{h(x+h+4)(x+4)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h+4)(x+4)} \\
&= \frac{-1}{(x+4)(x+4)} \\
&= \frac{-1}{(x+4)^2}.
\end{aligned}
$$

2. Several values are plotted for a function $y = f(x)$ below.
A. Numerically estimate the derivative of f at $x = 0, 2, 4, 6, 8, 10$. Wherever possible, your estimation should average the slopes from the left and right. Enter your answers in the spaces provided.
Answer: $f'(0) = 1.5$, $f'(2) = 1.25$, $f'(4) = 0.625$, $f'(6) = -0.25$, $f'(8) = -0.125$, $f'(10) = 0.5$.



B. Beginning with $f(0) = 0$, reconstruct $f(x)$ from the six derivatives you computed in Part A, using Euler's method with step size $\Delta x = 2$. This reconstructed f will not agree with the one above. Plot it on the same graph above, so that the discrepancy is clear.
Answer: We begin at $(0, 0$. $f'(0) = 1.5$, so the next y is $0 + 2 \cdot 1.5 = 3$. So the next point is $(2, 3)$. The next y is $3 + 2 \cdot 1.25 = 5.5$, so the next point is $(4, 5.5)$. Similarly, the remaining points are $(6, 6.75)$, $(8, 6.25)$, and $(10, 6.0)$. Graph these.

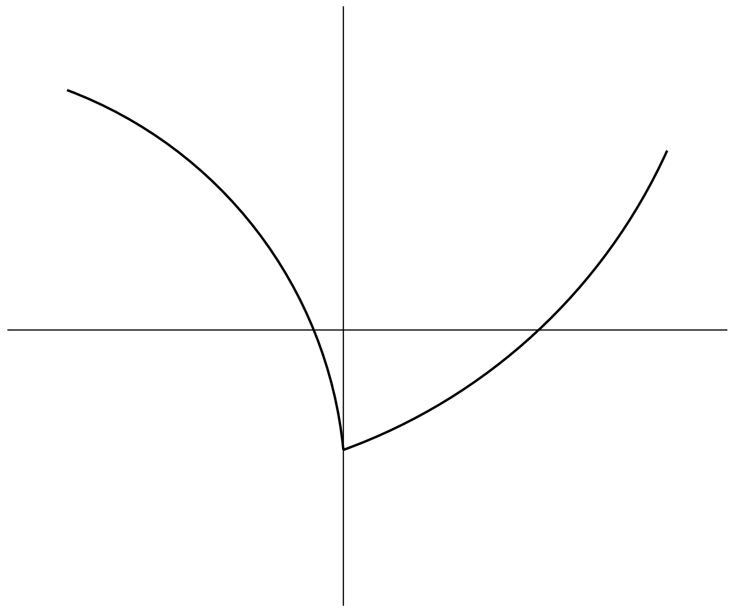
3. The graph of a function $y = f(x)$ is shown below. Sketch the graph of $y = f'(x)$ on top of it.
Answer:



4. Draw a graph $y = f(x)$ such that all three of these conditions are satisfied:

- $f'(x) < 0$ and $f''(x) < 0$ when $x < 0$,
- $f'(x) > 0$ and $f''(x) > 0$ when $x > 0$, and
- f is not differentiable at $x = 0$.

Answer:



5. Let $f(x) = e^{x^3-3x}$. Your answers to Parts C and D must be explained using algebra, rather than just graphing the function on your calculator.
A. Compute $f'(x)$.
Answer: $f'(x) = (e^{x^3-3x})' = e^{x^3-3x} \cdot (x^3 - 3x)' = e^{x^3-3x} \cdot (3x^2 - 3)$.
B. Compute $f''(x)$.
Answer:

$$\begin{aligned} f''(x) &= \left(e^{x^3-3x} \cdot (3x^2-3)\right)' \\ &= (e^{x^3-3x})' \cdot (3x^2-3) + e^{x^3-3x} \cdot (3x^2-3)' \\ &= e^{x^3-3x} \cdot (3x^2-3) \cdot (3x^2-3) + e^{x^3-3x} \cdot 6x \\ &= e^{x^3-3x} \cdot ((3x^2-3)^2 + 6x). \end{aligned}$$

C. For which x is $f(x)$ positive?
Answer: The function e^z is always positive, no matter what z is. So $f(x)$ is positive for all x.
D. For which x is $f(x)$ increasing?
Answer: We wish to find where $f'(x) = e^{x^3-3x} \cdot (3x^2 - 3)$ is positive. Since the first factor in $f'(x)$ is always positive, this boils down to finding where $3x^2 - 3$ is positive — in other words, where $x^2 > 1$. That's on the open set $(-\infty, 1) \cup (1, \infty)$.

6. A marshmallow is a cylinder of sugar-goo. One day you decide to cook one in a microwave oven. As it heats, it remains a cylinder, but it expands both in radius r and height h. I want to understand how the volume of the marshmallow changes.
A. Find a formula for the derivative of volume with respect to time, t.
Answer: We know $v = \pi r^2 h$, where both r and h depend on t. Using the product rule and the chain rule,

$$\begin{aligned} \frac{dv}{dt} &= \pi \frac{d}{dt}(r^2 h) \\ &= \pi \left(\frac{d}{dt}(r^2) \cdot h + r^2 \frac{d}{dt} h \right) \end{aligned}$$

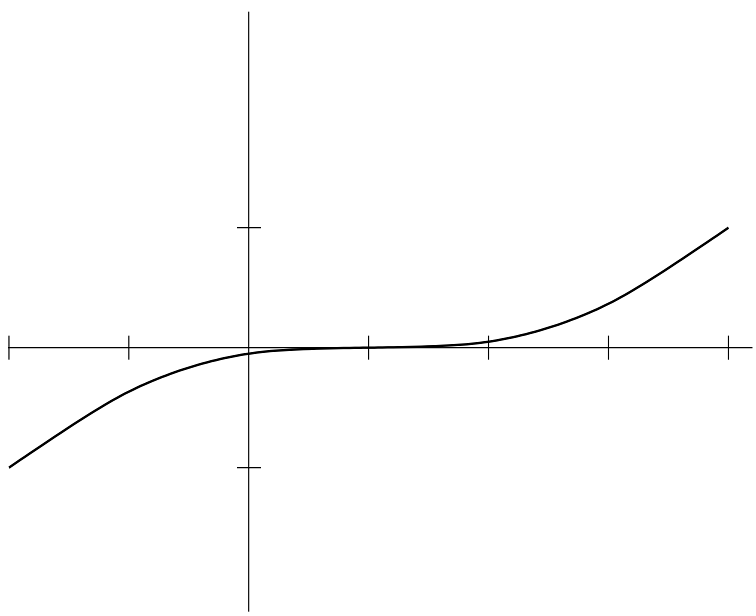
$$= \pi \left(2r\frac{dr}{dt} + r^2\frac{dh}{dt} \right).$$

B. You observe the following data as the marshmallow cooks. Use the data and the answer to Part A to compute how fast the volume is increasing at time $t = 5$. Include the appropriate units in your answer.

Time t (s)	5	10	15
Radius r (mm)	3	13	24
Height h (mm)	4	9	15

Answer: I estimate $dh/dt = (9 - 4)/(10 - 5) = 1$ and $dr/dt = (13 - 3)/(10 - 5) = 2$. Plugging these into the answer to Part A, along with $r = 3$ and $h = 4$, gives $dv/dt = 57\pi$.

7. Give a function $y = f(x)$ whose graph could be the one below.



Answer: $f(x) = (x - 1)^3/27$.

8. Compute the following limits, or explain why they do not exist.

A. $\lim_{x \to 0} \sqrt{x+2}\,(1 - e^x)$

Answer: Just plug in $x = 0$ to get a limit of 0.

B. $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$

Answer: Notice that $x^2 - 5x + 6 = (x - 2)(x - 3)$. So the function is question is really just $x - 3$. Now plug in $x = 2$ to get a limit of -1.

C. $\lim_{x \to \infty} \frac{x^2 + 2}{x(x - 1)(x + 7)}$

Answer: Expand the denominator. The highest-degree term is x^3. Divide the numerator and denominator by x^3. Then all terms in the numerator go to 0, while there is still a 1 in the denominator. So the limit is 0.