Math 31L-02 Spring 2007 Exam 2 Answers

1. Find $\lim_{x\to 0} \frac{\sin x}{x^{1/3}}$.
Answer:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x^{1/3}} &= \text{“}\frac{0}{0}\text{”} \\
&= \lim_{x\to 0} \frac{\cos x}{\frac{1}{3}x^{-2/3}} \qquad \text{(by L'Hôpital's Rule)} \\
&= \lim_{x\to 0} 3x^{2/3}\cos x \\
&= 3\cdot 0^{2/3}\cdot \cos 0 \\
&= 0.
\end{aligned}$$

2. This question is about the function $y = \arccos x$.
A. What is the definition of this function?
Answer: To say that $y = \arccos x$ is to say that $x = \cos y$ and y is in the interval $[0, \pi]$. That is, arccos is the inverse of cos restricted to this interval.
B. Find its derivative. (Even if you have the answer memorized, you must show how it is arises.)
Answer:

$$\begin{aligned}
\cos(\arccos(x)) &= x \\
\Rightarrow \quad -\sin(\arccos(x))\frac{d}{dx}(\arccos(x)) &= 1 \\
\Rightarrow \quad \frac{d}{dx}(\arccos(x)) &= \frac{-1}{\sin(\arccos(x))} \\
&= \frac{-1}{\sqrt{1-x^2}}.
\end{aligned}$$

(You can show the final equality using the Pythagorean theorem. If you don't know how, read the book's derivation of $\frac{d}{dx}(\arcsin(x))$ or talk to me.)

3. A conservation program is limiting the number of fish caught off the New Jersey coast. As a result, the fish have more time to mature and are larger when they're caught. Let $N(t)$ be the number of fish caught in year t of the program, and let $A(t)$ be the average mass of the fish caught in year t, in kg.
A. Let $M(t)$ be the total mass of fish caught. Give formulas for $M(t)$ and $M'(t)$.
Answer: $M(t) = N(t)A(t)$, and $M'(t) = N'(t)A(t) + N(t)A'(t)$, by the product rule.
B. In year 3 of the program, the average mass was 5 kg and increasing at a rate of 0.1 kg per year, and the number caught was $1,000,000$, decreasing at a rate of $100,000$ per year. What is $M'(3)$? Include units.
Answer: This says $A(3) = 5$, $A'(3) = 0.1$, $N(3) = 1,000,000$, and $N'(3) = -100,000$. Plug these into Part A to get $M'(3) = -400,000$ kg per year.
C. As less fish is caught, it becomes more valuable. Let $P(t) = \frac{1}{(M(t))^{3/2}}$ be the price in \$/kg, and let $R(t) = P(t)M(t)$ be the total revenue from selling fish in year t. Is $R(t)$ increasing or decreasing in year 3?

Answer: The problem says that $R = PM = M^{-1/2}$. So by the chain rule, $R'(t) = -\frac{1}{2}M^{-3/2}M'(t)$. At $t = 3$, we already know that $M'(3)$ is negative from Part B, and the $M^{-3/2}$ is positive, of course. So $R'(t)$ is positive and R is increasing.

4. (In this problem, the units of time are seconds, the units of distance are meters, the units of mass are kilograms, and the units of force are kg m/s^2.) A particle of mass m is travelling inside a particle accelerator. Its velocity at time $t = 0$ is 4. At time t, it experiences a force of $F(t) = 1.1^t$.
A. What is the acceleration $a(t)$ of the particle?
Answer: Newton's second law says $F = ma$. Therefore

$$a = \frac{F}{m} = \frac{1.1^t}{m}.$$

B. What is its velocity, $v(t)$?
Answer: We want to find a function $v(t)$ whose derivative is $a(t)$. Remember that m is a constant. If you know your exponential functions well and try out a couple of things, then you get

$$v(t) = \frac{1.1^t}{m \ln 1.1} + C.$$

(Check this answer!) Use the initial condition $v(0) = 4$ to get $C = 4 - \frac{1}{m \ln 1.1}$. Thus

$$v(t) = \frac{1.1^t}{m \ln 1.1} + 4 - \frac{1}{m \ln 1.1}.$$

C. At time t, how far is it from where it started?
Answer: Now we want to do the same thing to $v(t)$ to get distance $x(t)$, with $x(0) = 0$ since at time 0 it's obviously a distance of 0 from where it started. The answer is

$$x(t) = \frac{1.1^t}{m(\ln 1.1)^2} + \left(4 - \frac{1}{m \ln 1.1}\right) t - \frac{1}{m(\ln 1.1)^2}.$$

5. We have two reacting chemicals, A and B, with concentrations $[A] = [A](t)$ and $[B] = [B](t)$ governed by the differential equations

$$\frac{d[A]}{dt} = -k_1[A] + k_2[B],$$
$$\frac{d[B]}{dt} = k_1[A] - k_2[B].$$

Let $A_0 = [A](0)$ and $B_0 = [B](0)$. Assume $[A]$ is decreasing at all times t.
A. Since $[A]$ is decreasing at $t = 0$, what can we conclude about the constants A_0, B_0, k_1, and/or k_2? Give your answer as an inequality.
Answer: Since $[A]$ is decreasing, we know $\frac{d[A]}{dt} < 0$, so $-k_1[A] + k_2[B] < 0$, so $k_1[A] > k_2[B]$. This holds at all times t. In particular, at time $t = 0$, we have $k_1 A_0 > k_2 B_0$.
B. Prove that $[B]$ is always increasing.
Answer: Notice that $\frac{d[A]}{dt} + \frac{d[B]}{dt} = 0$, so the two derivatives are always opposite in sign. Since $\frac{d[A]}{dt} < 0$, that means $\frac{d[B]}{dt} > 0$, so $[B]$ is increasing.
C. Is $\dfrac{d^2[B]}{dt^2}$ always positive, always negative, or neither? Prove it.

Answer: Well,

$$\frac{d^2[B]}{dt^2} = \frac{d}{dt}\frac{d[B]}{dt} = \frac{d}{dt}(k_1[A] - k_2[B]) = k_1\frac{d[A]}{dt} - k_2\frac{d[B]}{dt}.$$

Since k_1 is positive and $\frac{d[A]}{dt}$ is negative, the first term is negative; since k_2 and $\frac{d[B]}{dt}$ are both positive, the second term is positive. A negative term minus a positive term must be negative. So $\frac{d^2[B]}{dt^2}$ is always negative.

D. When is $[B]$ increasing the fastest?

Answer: From Part C we know $\frac{d^2[B]}{dt^2}$ is always negative, which means that $\frac{d[B]}{dt}$ is always decreasing. So it is largest at the very start of the reaction, $t = 0$.

6. Let a be a positive constant, and consider the curve $x^2 + y^2 = a^2$.

A. Compute $\dfrac{dy}{dx}$.

Answer: Implicit differentiation gives $2x + 2y\frac{dy}{dx} = 0$, which simplifies to $\frac{dy}{dx} = -x/y$. (This is in the book, and we did it in class.)

B. Compute $\dfrac{d^2y}{dx^2}$, as simplified as you can make it.

Answer: Differentiating the answer from Part A using the quotient rule, we have

$$\frac{d^2y}{dx^2} = \frac{-1y - -x\frac{dy}{dx}}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-a^2}{y^3}.$$

C. At which points (x, y) is the curve concave down? Concave up? (For full credit, you must explain based on your answer to Part B.)

Answer: The top of the fraction is always negative. So when $y > 0$ the second derivative is negative and the curve is concave down; when $y < 0$ the second derivative is positive and the curve is concave up.