Name:

I have adhered to the Duke Community Standard.

Signature:

Math 31L 02 Spring 2007 Exam 3

Instructions: You have 50 minutes. You may use your TI-83 or equivalent calculator. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified answers, as exact as possible. Put a box around each answer. Ask questions if any problem is unclear. Good luck.

1. My company recreates antique rifles for U.S. Civil War reenacters. If I manufacture and sell q rifles, then my revenue is $R(q) = 500q - q^2$ and my cost is $C(q) = 150 + 10q$.
A. Write a function for my profit on q rifles.

B. How many rifles should I manufacture, to maximize profit?

2. This problems concerns the differential equation $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$. In the space below, sketch its slope field, and sketch the trajectory arising from initial condition (0, -2). (Your slope field should be large enough to show how the trajectory arises.)

2. True or False: Every continuous function has a continuous antiderivative. (Explain thoroughly.)

3. Thoroughly explain the distinction between these two objects:

$$\int_a^b f(x)\,dx \qquad \text{and} \qquad \int f(x)\,dx.$$

4. This problem concerns the differential equation $R'(x) = k(R^2 + 1)$, where k is a positive constant.

A. What are the equilibria of R?

B. Solve the differential equation.

C. How many initial conditions would I need to specify, for you to produce an answer with no unknown constants?

6. Imagine that inside a volcano is a vertical sheet of magma (molten rock). As the magma cools, the sheet gradually crystallizes (solidifies) from the . Heavy metallic elements suspended in the magma tend to end up in the magma that crystallizes first.
The city of Superville, Nebraska covers a circular region of radius 4 km. At a distance of x km from the city center, the population density is about

$$m(x) = \frac{30}{\sqrt{x}} \text{ people/km}^2.$$

A. Where is the population most dense? Where is it least dense?

B. Using n concentric rings, write a Riemann sum that approximates the total population of Superville.

C. Write an integral that represents the total population.

D. Evaluate that integral exactly.

E. What is the average population density of Superville, in people per km^2?

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Compute the following integrals. Give exact, simplified answers.

A. $\int_1^2 t^2 - 3t + 6\ dt$

B. $\int_0^{\pi/4} \frac{1}{1+x^2}\ dx$

2. Let $f(x) = \int_1^x (t-3)^2(e^t - 1)\ dt$. Find all local maxima and minima of $f(x)$. Show your work.

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4. You are the prime minister of Superland, a nation with a constant population of M people, some of whom are unfortunately infected with the Bad virus. Let $I(t)$ be the number of people with the virus on day t. The virus is mutating, which is causing its "infectiousness" (the ability to infect an exposed person) to change. Suppose that the rate of infection, in people per day, equals 10^{-7} times the product of the following three factors:

1. the number I of infected people,
2. the number of people not infected, and
3. the infectiousness, which your best scientists say is $10^{-6}(I - M/2)$.

A. Write a differential equation for the number of infected people.

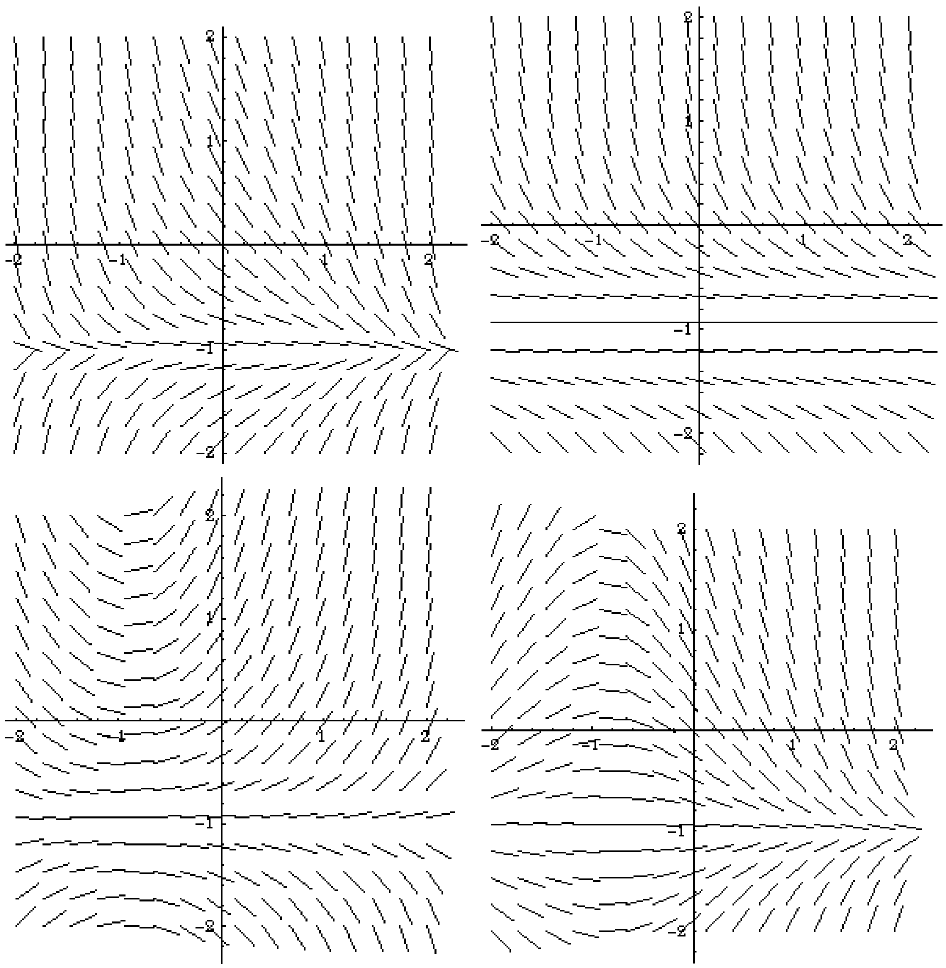
B. Find the equilibrium solutions of the differential equation.

C. After making a quick survey of the country, your health minister guesses that *roughly* half of the population is infected. From this, can you determine the long-term behavior of the infection? If so, what is it? If not, why not? (Hint: You may want to sketch a slope field.)

5. Label each slope field below with the letter (A, B, C, D) of the corresponding differential equation.

$$\text{A} : \frac{dy}{dx} = (-y-1)(1+x) \qquad \text{B} : \frac{dy}{dx} = (-y-1)(1+y)$$

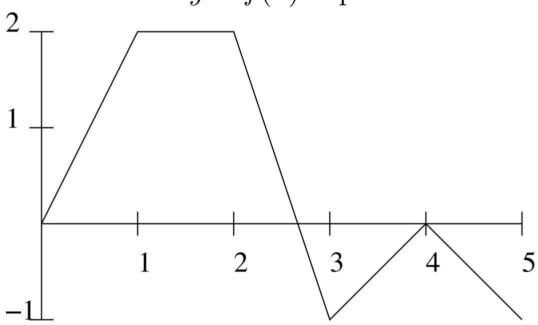
$$\text{C} : \frac{dy}{dx} = (-y-1)(1+x^2) \qquad \text{D} : \frac{dy}{dx} = (-y-1)(-1-x)$$



6. Clearly and thoroughly describe the distinction between

$$\int f(x)\,dx \quad \text{and} \quad \int_a^b f(x)\,dx.$$

7. A function $y = f(x)$ is pictured here:



A. On the picture, draw the area corresponding to the Riemann sum

$$\sum_{k=0}^{5} f\left(\frac{x_k + x_{k+1}}{2}\right)\Delta x,$$

where $\Delta x = 0.5$ and $x_k = 1 + k\Delta x$.

B. Let $F(x)$ be an antiderivative of $f(x)$. There are six pairs of numbers below. Between each pair, mark "<", ">", or "=" to indicate that relationship, or mark "?" to indicate that the relationship cannot be determined.

$\int_0^5 f(x)\,dx \qquad 0$ $\qquad\qquad$ $\int_0^3 f(x)\,dx \qquad 4$

$f'(0.5) \qquad F(2) - F(0)$ $\qquad\qquad$ $F(3) - F(1) \qquad 2$

$F(2) \qquad 2$ $\qquad\qquad$ $F(4) \qquad F(3)$

8. Does every continuous function have an antiderivative? Explain.

9. One fine winter day, the temperature of Lake Mendota is 7° (centigrade) while the ambient air temperature is -10°. Let $x(t)$ be the temperature of the lake on day t afterwards. Assume that the air stays at -10°, and that the lake cools at a rate equal to twice the difference in temperature between it and the air.
A. Write a differential equation and an initial condition to describe this problem.

B. Solve this initial value problem using separation of variables.