Math 31L 02 Spring 2007 Exam 3 Answers

Instructions: You have 50 minutes. You may use your TI-83 or equivalent calculator. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified answers, as exact as possible. Put a box around each answer. Ask questions if any problem is unclear. Good luck.

1. My company recreates antique rifles for U.S. Civil War reenacters. If I manufacture and sell q rifles, then my revenue is $R(q) = 500q - q^2$ and my cost is $C(q) = 150 + 10q$.

A. Write a function for my profit on q rifles.

Answer: The profit is $\pi(q) = R(q) - C(q) = -q^2 + 490q - 150$.

B. How many rifles should I manufacture, to maximize profit?

Answer: 245. [I'll omit the work, although of course you should not.]

2. Compute $\int_0^1 \sin x + 14x \, dx$ exactly.

Answer:

$$\begin{aligned}
\int_0^1 \sin x + 14x \, dx &= \Big[-\cos x + 7x^2\Big]_0^1 \\
&= (-\cos 1 + 7) - (-\cos 0 + 0) \\
&= -\cos 1 + 8.
\end{aligned}$$

3. Inside a volcano there is a chamber of molten hot *magma* (liquid rock). The magma chamber is a vertical box, 1 km wide in the east-west direction, 2 km long in the north-south direction, and extending from the surface down to a depth of 6 km. Years pass and the magma cools into solid rock. Due to a process known as *fractional crystallization*, the rock that forms at the bottom of the magma chamber is denser than the rock at the top. The density at depth y is

$$f(y) = 2.6e^{0.04y}$$

(in units of mass per km^3).

A. Write a Riemann sum of n terms that approximates the total mass of the rock formed by the magma chamber. Explain.

Answer: Divide the chamber into n horizontal slabs of thickness $\Delta y = 6/n$. For $k = 0, 1, \ldots, n-1$, the kth slab runs from depth $y_k = k\Delta y$ to depth $y_{k+1} = (k+1)\Delta y$. The density throughout this slab is approximately $f(y_k) = 2.6e^{0.04y_k}$. So the mass of the slab is about

$$2.6e^{0.04y_k} \cdot 2 \cdot 1 \cdot \Delta y = 5.2e^{0.04y_k} \Delta y,$$

and the total mass of the rock formed by the magma chamber is about

$$\sum_{k=0}^{n-1} 5.2e^{0.04y_k} \Delta y.$$

B. Write the corresponding definite integral, and explain how the definite integral is related to the Riemann sum.

Answer: The definite integral is

$$\int_0^6 5.2e^{0.04y} \, dy = \lim_{n\to\infty} \sum_{k=0}^{n-1} 5.2e^{0.04y_k} \Delta y.$$

C. Compute the integral.
Answer:

$$\int_0^6 5.2e^{0.04y}\,dy = \left[\frac{5.2}{0.04}e^{0.04y}\right]_0^6 = \frac{5.2}{0.04}\left(e^{0.24}-1\right).$$

4. True or False: Every continuous function has a continuous antiderivative. (Explain thoroughly.)
Answer: True. Given any continuous function $f(x)$, let $F(x) = \int_0^x f(t)\,dt$. Then the second fundamental theorem of calculus says that the derivative of this function $F(x)$ is $f(x)$. Furthermore, $F(x)$ must be continuous, since it is differentiable. In summary, $f(x)$ has a continuous antiderivative, namely the $F(x)$ defined here. (Of course, by adding a constant C to $F(x)$ we can find the other antideriatives of $f(x)$.)

5. Thoroughly explain the distinction between these two objects:

$$\int_a^b f(x)\,dx \qquad \text{and} \qquad \int f(x)\,dx.$$

Answer: The object on the left is the (definite) integral of f from $x = a$ to $x = b$. It is a *number*, equal to the (signed) area of the region trapped by the graph of $y = f(x)$. It arises as a limit of sums of areas of rectangles that approximate this region. In contrast, the object on the right is the antiderivative of f, sometimes called the indefinite integral. It is a *set of functions*, namely all of the functions whose derivative is f. It turns out that these functions are all closely related — they all differ from each other by the addition of some constant C.

6. On west campus there are many rodents, and there are also many snakes eating those rodents. After a careful study, I suspect that

$$\begin{aligned}\frac{dR}{dt} &= kR - \ell S,\\ \frac{dS}{dt} &= mS,\end{aligned}$$

where $R(t)$ is the number of rodents at time t, $S(t)$ is the number of snakes at time t, and k, ℓ, and m are positive constants. I'd like to solve these differential equations.
A. In words, explain why each of the terms in these differential equations makes sense.
Answer: In the second equation, the mS term says that the snake population is always growing at a rate proportional to its current size; this is the simplest model of population growth. In the first equation, the kR term indicates that the rodent population "wants" to grow in the same way, but the $-\ell S$ term indicates that rodents die in some amount proportional to the number of snakes present, presumably because lots of snakes eat lots of rodents.
B. Solve the second differential equation for $S(t)$.
Answer: $S(t) = Ae^{mt}$, where A is an unknown constant.
C. Using your solution to Part B, rewrite the first differential equation so that S does not appear.
Answer: $\frac{dR}{dt} = kR - \ell Ae^{mt}$.
D. I'd like to solve the differential equation from Part C. Name all of the methods we've learned this semester that would be appropriate for this equation. (You do not have to solve the equation.)
Answer: The differential equation from Part C does not appear to be separable. Since the method of separation subsumes our earlier methods (recognizing exponential growth, z-substitution,

etc.), none of them can work. All that we can do is find an approximate solution using Euler's method or a slope field.

7. Consider the differential equation $R'(x) = k(R^2 + 1)$, where k is a positive constant.
A. Solve the differential equation.
Answer: Writing $R'(x) = dR/dx$, we can separate the equation to

$$\frac{1}{R^2+1}\, dR = k\, dx.$$

Then $\int k\, dx = kx + C$, and

$$\int \frac{1}{R^2+1}\, dR = \arctan R.$$

(We can omit the "$+C$" here.) Thus $\arctan R = kx + C$, so $R = \tan(kx + C)$.
B. How many initial conditions would I need to specify, for you to produce an answer with no undetermined constants?
Answer: If you were given neither k nor C, then you would need two initial conditions to find them. If you were given k, then you would need only one initial condition to find C.

8. This problems concerns the differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$. In the space below, sketch its slope field, and sketch the trajectory arising from initial condition (0, -2). (Your slope field should be large enough to show the whole trajectory.)
Answer: [The slope field looks like Figure 11.14 in your book, but reflected across the y-axis. The trajectory that starts at $(0, -2)$ spirals into the origin in a counterclockwise manner.