Carleton College CS 202, Fall 2008, Final Exam

You have 150 minutes.

You may not use a calculator. You may use a standard-size ($8.5'' \times 11''$) note sheet with notes written or typed on both sides *by you*.

Except on the TRUE/FALSE/PUNT questions, always show your work and explain all of your answers. Good work often earns partial credit. A correct answer with no explanation often earns little or no credit.

If you have no idea how to solve a problem, or if you have forgotten a key formula that you think you need to know, then you may ask me for a hint. The hint will cost you some points (to be decided by me as I grade your paper), but will probably help you earn more points overall.

Good luck.

1. I have 13 distinct programs (call them $A, B, \ldots, M$) to run on 3 computers (call them $1, 2, 3$). The first computer has 1 processor, the second 4 processors, and the third 8 processors. I want to distribute the programs among the computers so that on each computer the number of programs equals the number of processors. How many ways are there to do this?

2. In *The Merchant of Venice* William Shakespeare writes, "All that glitters is not gold." Is that statement true? Discuss.

3. (Note: The purpose of this problem is to verify that you can write a proof by induction. So write it well.) Prove that for any integer $n \geq 0$, $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

4. A. Find all solutions to the recurrence relation $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$.

B. Find the solution that satisfies the initial conditions $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

5. Each part of this problem is a true/false question, but there are three valid answers: TRUE, FALSE, and PUNT. If you answer PUNT, then you receive half credit. Otherwise, if you answer correctly then you receive full credit, and if you answer incorrectly then you receive no credit. No explanation is necessary on this problem. Do not just write T, F, or P; write the entire word.

A. Among the positive integers, $\forall a\ \forall b\ \forall c\ (a|bc \rightarrow (a|b \vee a|c))$.

B. Among the positive integers, $\forall a\ \forall m\ \exists b\ ab \equiv \gcd(a, m) \pmod{m}$.

C. Among general functions $f : X \rightarrow Y$, $f(f^{-1}(y)) = \{y\}$.

D. Among general functions $f : X \rightarrow Y$, $f^{-1}(f(x)) = \{x\}$.

E. If X and Y are finite sets, then $|X \times Y| = |X| + |Y|$.

F. The function $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(m, n) = m - n$ is surjective (onto).

G. The function $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(m, n) = m - n$ is injective (one-to-one).

H. The number of 32-bit bitstrings with 16 or more 1s equals the number of 32-bit bitstrings with 16 or fewer 1s.

6. My friend and I send and receive 8-bit messages on a noisy channel. In transit, each of the 8 bits in a message has a 1% chance of being flipped. (These probabilities are independent.) So we use a parity system. Every valid message contains an even number of 1s; if I receive a message with an odd number of 1s, then I know that at least one error has occurred.

My friend sends me the 8-bit message 00000000. Given that I receive a message with an even number of 1s, what is the probability that no errors occurred in transmission?

7. In this problem, graphs are directed, with weighted edges (with positive real weights), with no vertex loops and no multi-edges. Graphs are stored as adjacency lists; accessing a list element is $\mathcal{O}(n)$. I will now describe Dijkstra's algorithm for finding shortest paths through such a graph $G = (V, E)$ from a given vertex $s \in V$. Each vertex has (1) a color, initially set to green, (2) a distance, initially set to ∞, and (3) a predecessor vertex, initially set to `None`. For the starting vertex s, set its distance to 0. Then perform the following loop until all vertices are red.

Find the green vertex v that has the least distance. (Break ties arbitrarily.) For each neighboring vertex w (i.e. there exists an edge from v to w), compute the sum of v's distance and the weight of the edge from v to w. If this number is less than w's distance, then update w's distance to it and set w's predecessor to v. Once this has been done for all of v's neighbors, color v red.

A. Execute Dijkstra's algorithm on the following graph.

B. In general, what is the running time, in terms of $|V|$ and $|E|$? Use big-$\mathcal{O}$ notation.

8. A chocolate bar consists of n squares arranged in a rectangular grid. It can be broken along any horizontal or vertical grid line into two smaller bars. Your goal is to break the bar repeatedly, until there are n separate squares. Assuming that only one bar can be broken at a time, how many breaks must you make? Prove your answer using strong induction.

9. In this multi-part problem, graphs are undirected (so that the edge (v, w) is identical to the edge (w, v)), with no vertex loops and no multi-edges. For any integer $k \geq 0$, a graph $G = (V, E)$ is said to be *k-regular* if every vertex has degree k. For $k \geq 0$ and $n \geq 0$, let $P(k, n)$ be the statement "There exists a k-regular graph with n vertices."

A. Prove that $\forall$ odd k $\forall$ odd n, $\neg P(k, n)$.

This space is for scratch work.

[Remark: When this exam was given, the students were directed to ignore page 10.]

B. Prove *one* of the following statements, of your choice.

1. $\forall$ odd k $\forall$ even $n > k$, $P(k, n)$.
2. $\forall$ even k $\forall n > k$, $P(k, n)$.