1. In each part, compute dx/dt.

A. $x = \log_7\left(\tan\left(3^t\right)\right)$.

Answer:

$$\frac{dx}{dt} = \frac{1}{\tan\left(3^t\right)\ln 7} \cdot \sec^2\left(3^t\right) \cdot 3^t \ln 3.$$

B. $x^3 + t^3 = \sin x$.

Answer: Implicitly differentiating, we get

$$\begin{aligned} 3x^2 \cdot \frac{dx}{dt} + 3t^2 &= \cos x \cdot \frac{dx}{dt} \\ \Rightarrow \quad 3x^2 \cdot \frac{dx}{dt} - \cos x \cdot \frac{dx}{dt} &= -3t^2 \\ \Rightarrow \quad \frac{dx}{dt} &= \frac{-3t^2}{3x^2 - \cos x}. \end{aligned}$$

C. $x = \left(\sqrt{t}\right)^{\ln t}$.

Answer: Using the technique of logarithmic differentiation,

$$\begin{aligned} \ln x &= \ln\left(\left(\sqrt{t}\right)^{\ln t}\right) \\ &= \ln t \cdot \ln\left(\sqrt{t}\right) \\ \Rightarrow \quad \frac{1}{x}\frac{dx}{dt} &= \frac{1}{t} \cdot \ln\left(\sqrt{t}\right) + \ln t \cdot \frac{1}{\sqrt{t}} \cdot \frac{1}{2}t^{-1/2} \\ \Rightarrow \quad \frac{dx}{dt} &= x \cdot \left(\frac{1}{t} \cdot \ln\left(\sqrt{t}\right) + \ln t \cdot \frac{1}{\sqrt{t}} \cdot \frac{1}{2}t^{-1/2}\right) \\ &= \left(\sqrt{t}\right)^{\ln t} \cdot \left(\frac{1}{t} \cdot \ln\left(\sqrt{t}\right) + \ln t \cdot \frac{1}{\sqrt{t}} \cdot \frac{1}{2}t^{-1/2}\right). \end{aligned}$$

2. Antidifferentiate.

A. $\displaystyle\int 3s^7 - 2\sqrt{s} + 1\; ds$

Answer:

$$\int 3s^7 - 2\sqrt{s} + 1\; ds = \frac{3}{8}s^8 - \frac{4}{3}s^{3/2} + s + C.$$

B. $\displaystyle\int \frac{y^4}{y^5 + 4}\; dy$

Answer: Letting $u = y^5 + 4$ eventually yields

$$\int \frac{y^4}{y^5 + 4}\; dy = \frac{1}{5}\ln(y^5 + 4) + C.$$

C. $\displaystyle\int \frac{4^{\ln(t^2)}}{t}\; dt$

Answer: Letting $u = \ln(t^2)$ eventually yields

$$\int \frac{4^{\ln(t^2)}}{t}\, dt = \frac{1}{2\ln 4} 4^{\ln(t^2)} + C.$$

3. You are lying on your back in cool grass on a warm summer day. In the sky you see a delightful cloud that resembles a fluffy bunny. The cloud is at an altitude of 2 km and traveling 30 km/h horizontally. When it is directly overhead, how fast must your eyes rotate, to keep staring at it? Include units in your answer.

Answer: Let t be time (in hours), x the horizontal distance (in kilometers) between the point 2 km directly overhead and the cloud, and θ the angle (in radians, of course) made by the vertical line and the line-of-sight to the cloud. [A picture helps, but I'll omit it.] We know that $dx/dt = 30$ km/h; we wish to compute $d\theta/dt$. From basic trigonometry,

$$\tan\theta = \frac{x}{2}.$$

Differentiating with respect to t, we get

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}.$$

Plugging in $\theta = 0$ (indicating that the cloud is directly overhead) and $dx/dt = 30$ yields

$$\frac{d\theta}{dt} = 15.$$

The units are radians per hour.

4. Your company makes tiny plastic figurines of toads. If you make and sell x shipments (of one million toads each), then your manufacturing cost per shipment is

$$m(x) = 3 + \frac{1}{x^{1.9}}$$

(millions of dollars), and your sale price per shipment is

$$s(x) = 3 + \frac{2}{x^{1.3}}$$

(millions of dollars). Find a function $p(x)$ that describes the profit (income minus costs) that you earn from x shipments of toads; then maximize that function on an appropriate interval.

Answer: For x shipments, the total income is $xs(x) = 3x + 2x^{-0.3}$ and the total cost is $xm(x) = 3x + x^{-0.9}$. Therefore the profit for x shipments is

$$p(x) = xs(x) - xm(x) = 2x^{-0.3} - x^{-0.9}.$$

Only $x \geq 0$ make sense for the problem, and the function is undefined at $x = 0$ (apparently the model breaks down there), so we will maximize $p(x)$ on the interval $(0, \infty)$. The derivative is

$$p'(x) = -0.6x^{-1.3} + 0.9x^{-1.9}.$$

This is defined on all of $(0, \infty)$. It is zero when

$$\begin{aligned} & -0.6x^{-1.3} + 0.9x^{-1.9} &=& \ 0 \\ \Rightarrow \quad & -0.6x^{0.6} + 0.9 &=& \ 0 \\ \Rightarrow \quad & x^{0.6} &=& \ \frac{-0.9}{-0.6} \\ & &=& \ \frac{3}{2} \\ \Rightarrow \quad & x &=& \ \left(\frac{3}{2}\right)^{5/3} \\ & &\approx& \ 1.96556. \end{aligned}$$

The second derivative of profit is

$$p''(x) = (0.6)(1.3)x^{-2.3} - (0.9)(1.9)x^{-2.9}.$$

So $p(x)$ is concave-down when

$$\begin{aligned} & (0.6)(1.3)x^{-2.3} - (0.9)(1.9)x^{-2.9} &<& \ 0 \\ \Leftrightarrow \quad & (0.6)(1.3)x^{0.6} - (0.9)(1.9) &<& \ 0 \\ \Leftrightarrow \quad & x^{0.6} &<& \ \frac{(0.9)(1.9)}{(0.6)(1.3)} \\ \Leftrightarrow \quad & x &<& \ \left(\frac{(0.9)(1.9)}{(0.6)(1.3)}\right)^{5/3} \\ & &\approx& \ 3.69972. \end{aligned}$$

Therefore $p(x)$ is concave-down at the critical point, so that critical point is a local maximum. It is the only critical point on $(0, \infty)$, so it must be the global maximum. The maximum profit is therefore about

$$p(1.96556) \approx 1.08866.$$

[Remark: Many students wrote $p(x) = s(x) - m(x)$ instead of $p(x) = xs(x) - xm(x)$. I tried to accommodate this error in my grading, by marking off only two points for it. Students missed many points for routine, mechanical aspects of optimization: naming an interval, checking not only where the derivative is zero but also where it is undefined, and checking that the critical point found is really a maximum (as opposed to a minimum or something else).]

5. A. Given a number $a \neq 0$, describe a procedure based on Newton's method that lets you compute $1/a$ without using long division. Simplify if you can.

Answer: We are trying to find $x = 1/a$, which is equivalent to $a = 1/x$, which is equivalent to $1/x - a = 0$. So let

$$f(x) = x^{-1} - a;$$

we wish to find a zero of $f(x)$. The derivative is $f'(x) = -x^{-2}$, and the Newton's method iteration formula is

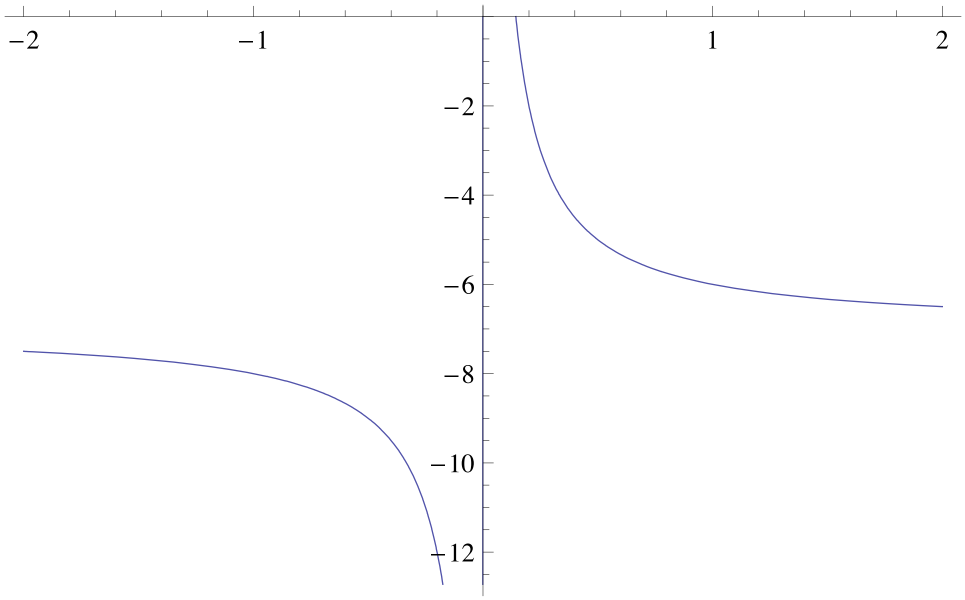
$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{-1} - a}{-x_n^{-2}} \\ &= x_n + x_n^2(x_n^{-1} - a) \\ &= x_n + x_n - ax_n^2 \\ &= 2x_n - ax_n^2. \end{aligned}$$

To find a zero of $f(x)$ (which is equivalent to computing $1/a$), we pick a seed value x_1 and then repeatedly apply the iteration formula to obtain x_2, x_3, and so on. When successive x_i agree to the desired number of decimal places, we can stop. Notice that nowhere in this iterative process do we ever divide anything by a!

[Remark: This was taken directly from 4.8 #30, which was assigned as homework.]

B. Use Part A to compute $1/7$ with a starting value of $x_1 = 1$. What happens? Why?

Answer: Starting with $x_1 = 1$, we get $x_2 = -5$, $x_3 = -185$, and $x_4 = -239945$. These numbers do not seem to be converging to $1/7$. An examination of the graph of $f(x)$ (below) reveals the reason. The tangent line at $x_1 = 1$ intersects the x-axis far to the left of 0, namely at $x_2 = -5$. The tangent line at $x_2 = -5$ intersects the x-axis even farther to the left. This pattern continues; each step takes us farther away from the zero just to the right of 0.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