1. Differentiate the following functions. Do NOT simplify.

A. $y = 5x^6 - 2x^3 + 3x + 1$.

$$\frac{dy}{dx} = 30x^5 - 6x^2 + 3.$$

B. $s(t) = \dfrac{t}{\cos t}$.

$$\frac{ds}{dt} = \frac{\cos t + t \sin t}{\cos^2 t}.$$

C. $p = 4^r \cdot (3r - 1)$.

$$\frac{dp}{dr} = 4^r \ln 4 \cdot (3r - 1) + 4^r \cdot 3.$$

D. $y = \left(2e^{5t} + \sin\left(\sqrt{t}\right)\right)^7$.

$$\frac{dy}{dt} = 7\left(2e^{5t} + \sin\left(\sqrt{t}\right)\right)^6 \cdot \left(2e^{5t} \cdot 5 + \cos\left(\sqrt{t}\right) \cdot \frac{1}{2}t^{-1/2}\right).$$

2. The Furtwängler Glacier on Mount Kilimanjaro is a giant sheet of ice. Suppose, for simplicity, that it is rectangular of length 600 m and width 100 m. Due to warming climate, its length and width are both shrinking by 5 m per year. Its thickness is a constant 6 m. How fast is the volume of the glacier changing? Simplify your answer, and include units.

Since the volume v is the product of the thickness 6, length ℓ, and width w, we have $v = 6\ell w$. Let t be time in years. Then $\frac{d\ell}{dt} = \frac{dw}{dt} = -5$, and

$$\begin{aligned}
\frac{dv}{dt} &= \frac{d}{dt}(6\ell w) \\
&= 6\left(\frac{d\ell}{dt}w + \ell\frac{dw}{dt}\right) \\
&= 6(-5 \cdot 100 + 600 \cdot -5) \\
&= -30 \cdot 700 \\
&= -21000.
\end{aligned}$$

Thus the glacier is losing 21000 m^3 of volume per year.

3. Suppose that $x = f(t) = \cos(2t) + \sin(3t)$ is the position of a particle at time t. Compute the particle's acceleration. Simplify and clearly mark your answer.

The velocity is $f'(t) = -2\sin(2t) + 3\cos(3t)$. The acceleration is $f''(t) = -4\cos(2t) - 9\sin(3t)$.

4. Recall that Newton's law of cooling is expressed by the differential equation $\dfrac{dy}{dt} = k(A - y)$.

A. Explain in words the physical meaning of these five quantities:

t is time.

y is the temperature of the cooling (or warming) body.

A is the ambient temperature of the environment, assumed to be constant.

k is a constant of proportionality with units 1 / time.

$\frac{dy}{dt}$ is the rate of change of the body temperature with respect to time.

B. We used $k = 0.9$ in the Santa-cooling problem in class. If we had used a lesser (but still positive) value of k, such as 0.5, would Santa cool more slowly or more rapidly? Explain.

If k were made smaller, then $\frac{dy}{dt}$ would become smaller, meaning that the rate of change of Santa's temperature would be smaller. So he would cool more slowly.

C. The value $k = 0.9$ was given to us by the police but not explained further. In words, describe some circumstances of Santa's death that might influence the police estimate for k.

What circumstances might affect the rate at which Santa cools (other than the ambient temperature, which is already described by A)? It comes down to various forms of insulation: body fat, clothing, whether he was indoors (in a house or car) or outdoors, how much wind there was, etc.

5. Show that $\lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$.

Since $\sin x$ is always between -1 and 1, so is $\sin(1/x)$. It follows that $x\sin(1/x)$ is always between $-|x|$ and $|x|$. (To convince yourself, you can plot $y = x$, $y = -x$, and $y = x\sin(1/x)$ all on one graph.) Thus for all $h \neq 0$,

$$-|h| \leq h\sin(1/h) \leq |h|.$$

Now $-|h|$ and $|h|$ both have limit 0 as $h \to 0$, and $h\sin(1/h)$ is trapped between them, so the Squeeze Theorem says that $h\sin(1/h)$ must also have limit 0 as $h \to 0$. (This problem was taken from our homework; see Problem 6 below.)

6. Compute the derivative, at $x = 0$, of $y = f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

We use the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} \\ &= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) \\ &= 0, \end{aligned}$$

by Problem 5 above. (This problem was taken from our homework, namely 2.7 #52.)

7. The table below shows the number s of web sites in existence, for each year t in a 10-year period. I wish to model s as a function of t. When I plot the data on a semilog plot, I get a line of slope 1.43 and intercept 2.31. What then is the function $s(t)$? Simplify your answer.

year t	1	2	3	4	5	6	7	8	9	10
# sites s	42	176	735	3070	12800	53600	224000	936000	3910000	16400000

The semilog plot relates t to $\ln s$, so the equation of the line is $\ln s = 1.43t + 2.31$. Exponentiating both sides of that equation, we get

$$s = e^{1.43t+2.31} = e^{2.31}e^{1.43t}.$$

8. Graph $y = e^{2x} - 4$ as precisely as you can (including the correct scale, intercepts, etc.).

The graph is like that of $y = e^x$, but compressed by a factor of 2 horizontally and shifted down 4. The y-intercept is at $(0, -3)$. The x-intercept is where

$$\begin{array}{rrcl} & e^{2x} - 4 & = & 0 \\ \Leftrightarrow & (e^x)^2 & = & 4 \\ \Leftrightarrow & e^x & = & \pm 2 \\ \Leftrightarrow & e^x & = & 2 \\ \Leftrightarrow & x & = & \ln 2. \end{array}$$

So it's at $(\ln 2, 0)$. I'll omit the graph from these solutions; you can graph it for yourself using *Mathematica* or a graphing calculator.