1. Compute.

A. $\int t^{17}\,dt = \frac{1}{18}t^18 + C.$

B. $\int x\sin(3x^2)\,dx = \frac{-1}{6}\cos(3x^2) + C.$

C. $\int \frac{e^{\sqrt{u}}}{\sqrt{u}}\,du = 2e^{\sqrt{u}} + C.$

2. Differentiate these functions.

A. $s(t) = \ln(\tan(t^2)) \Rightarrow \frac{d}{dt}s(t) = \frac{1}{\tan(t^2)} \cdot \sec^2(t^2) \cdot 2t.$

B. $f(x) = x^{(e^{2x}-1)}$? Let $y = x^{(e^{2x}-1)}$, so $\ln y = \ln(x^{(e^{2x}-1)}) = (e^{2x}-1)\ln x$. Differentiating the equation with respect to x yields

$$\frac{1}{y}y' = e^{2x} \cdot 2 \cdot \ln x + (e^{2x}-1) \cdot \frac{1}{x},$$

which implies that

$$\begin{aligned} y' &= y\left(e^{2x} \cdot 2 \cdot \ln x + (e^{2x}-1) \cdot \frac{1}{x}\right) \\ &= x^{(e^{2x}-1)}\left(e^{2x} \cdot 2 \cdot \ln x + (e^{2x}-1) \cdot \frac{1}{x}\right). \end{aligned}$$

3. After a lot of practice, I've become good at estimating e^a for any number a. But still I have trouble computing $\ln a$. Describe in detail a procedure, based on Newton's method, that I can use to estimate $\ln a$ for any positive number a.

Solving $x = \ln a$ is tantamount to solving $e^x = a$ or $e^x - a = 0$. So let $f(x) = e^x - a$; we wish to find a zero of $f(x)$. The Newton's method iteration formula is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{e^{x_n} - a}{e^{x_n}} \\ &= x_n - 1 + ae^{-x_n}. \end{aligned}$$

Given $a > 0$, estimate $\ln a$ using the following procedure. First select a seed value x_1. Plug this into the iteration formula to get x_2. Plug that into the iteration formula to get x_3. Continue in this manner until successive x_n agree to the desired number of decimal places.

4. You are trying to persuade the government of British Columbia to purchase a $10,000$ km^2 patch of land from the various people who currently own parts of it, to set it aside as a wildlife habitat. The more of it the government purchases, the better, because wild animals prefer large tracts of land far away from human influence; your research suggests that a preserve of x km^2 could sustain $m(x) = \frac{1}{100}x^2$ large mammals. Unfortunately, the more land the government purchases, the more expensive each additional km^2 becomes; it seems that the cost of purchasing

x km^2 will be about $c(x) = \frac{25}{10{,}000-x}$ (in thousands of Canadian dollars). How much land should the government buy, to maximize *the number of large mammals protected per dollar*?

If the government buys x km^2, then the number of large mammals protected per thousand dollars is

$$f(x) = \frac{m(x)}{c(x)} = \frac{x^2(10000-x)}{100 \cdot 25} = 4x^2 - \frac{1}{2500}x^3.$$

We wish to maximize $f(x)$ for x in the interval $[0, 10000]$. The derivative is

$$f'(x) = 8x - \frac{3}{2500}x^2 = x\left(8 - \frac{3}{2500}x\right).$$

The derivative is never undefined. It is 0 when $x = 0$ and when $8 - \frac{3}{2500}x = 0$, which is when $x = 8 \cdot \frac{2500}{3} = \frac{20000}{3}$. In summary, here is a table giving all critical points of $f(x)$ and endpoints of $[0, 10000]$, and the values of $f(x)$ at those points:

x	0	$\frac{20000}{3}$	10000
$f(x)$	0	1600000000/27	0

So the greatest large animals protected per thousand dollars occurs when the government purchases $20000/3$ km^2 ≈ 6667 km^2 of land; it protects $1600000000/27 \approx 5.9 \cdot 10^7$ large animals per thousand dollars then, or $1600000/27 \approx 5.9 \cdot 10^4$ large animals per dollar.

Remark: This $5.9 \cdot 10^4$ figure is too large to be realistic; sorry.

5. Let $B = \displaystyle\int_{-1}^{7} \sin(t^2)\,dt$.

A. What kind of object is B (e.g. a number, a function, a region in the plane, a point)? What is its geometric meaning?

B is a number. It is the exact signed area trapped among the curves $y = \sin(t^2)$, $y = 0$, $t = -1$, and $t = 7$. By "signed area" I mean that area below the t-axis is counted negatively in the integral.

B. Express B as a limit of Riemann sums. Be as detailed as possible.

In the standard notation, we have $a = -1$, $b = 7$, $\Delta t = \frac{b-a}{n} = \frac{8}{n}$,

$$t_i = a + i\Delta t = -1 + i\frac{8}{n},$$

and $f(t) = \sin(t^2)$. Therefore, using right-hand sums,

$$\int_{-1}^{7} \sin(t^2)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\left(-1 + i\frac{8}{n}\right)^2\right)\frac{8}{n}.$$

C. Explain how you would find an approximate value for B, if you were given a calculator.

Given a calculator, I would pick a large value for n — perhaps $n = 1000$ — and evaluate the Riemann sum for that n:

$$B \approx \sum_{i=1}^{n} \sin\left(\left(-1 + i\frac{8}{n}\right)^2\right)\frac{8}{n}.$$

6. In as much detail as possible, graph $y = xe^x$.

For this problem, it is helpful to keep in mind that e^x is always positive.

Notice that y is defined everywhere and zero only at $x = 0$; it is negative for $x < 0$ and positive for $x > 0$. The derivative is

$$y' = e^x + xe^x = (x+1)e^x,$$

which is defined everywhere and zero only at $x = -1$. It is negative for $x < -1$ (so y is decreasing there) and positive for $x > -1$ (so y is increasing there). The second derivative is

$$y'' = e^x + (x+1)e^x = (x+2)e^x,$$

which is defined everywhere and zero only at $x = -2$. It is negative for $x < -2$ (so y is concave-down there) and positive for $x > -2$ (so y is concave-up there).

The concavity changes at $x = -2$, so that is an inflection point. The function changes from decreasing to increasing at $x = -1$, so that is a local minimum. (Alternatively, $x = -1$ is a critical point with $y'' > 0$, so it is a local minimum.)

There are no vertical asymptotes, since y is always defined. To find the horizontal asymptotes, we take limits as $x \to \pm\infty$. The limit as $x \to \infty$ is $+\infty$. The limit as $x \to -\infty$ is difficult to compute with our current knowledge in the class. (We haven't studied L'Hopital's Rule or power series.) However, based on the fact that $y < 0$ and $y' < 0$ for $x < -1$, we know that y must stay between 0 and $y(-1) = \frac{-1}{e}$ for all $x < -1$.

Despite not knowing the limit as $x \to -\infty$, we can create a fairly detailed graph from what we do know. To see it, examine Section 4.5 Example 3, on page 311.