Carleton College Math 232, Spring 2008, Exam 1

You have 70 minutes.

You may not use any notes or calculator.

Always show your work and explain all of your answers. Good work often earns partial credit. A correct answer with no explanation often earns little or no credit.

If you have no idea how to solve a problem, or if you have forgotten a key formula that you think you need to know, then you may ask me for a hint. The hint will cost you some points (to be decided by me as I grade your paper), but will probably help you earn more points overall.

Good luck.

1. Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 4 & 4 \\ -1 & 1 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 14 \\ 0 \end{bmatrix}, \qquad \vec{c} = \begin{bmatrix} 14 \\ 0 \\ 14 \end{bmatrix}.$$

Compute A^{-1} and use it to solve $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{c}$. Clearly mark your answers.

2. Find the polynomial $y = ax^3 + bx^2 + cx + d$ that passes through the points $(-1, 3)$, $(0, 1)$, $(1, 3)$, and $(2, 4)$. (Note: Most of the points for this problem are earned by setting up the relevant equations. Only solve the equations if you have extra time at the end of the exam.)

3. In the Democratic Republic of Pretendland (DRP) there are n major cities, which are called $C_1, \ldots, C_n$ (in decreasing order of population). Some of the cities are connected by non-stop high-speed trains; let A be the $n \times n$ adjacency matrix of this train network. Some of the cities are connected by non-stop airplane flights; let B be the $n \times n$ adjacency matrix of this airplane network. Just so we're clear: There is one set of cities here, but there are two separate graphs built on them, with different adjacency matrices.

A. You are planning a trip to the DRP in which you'll be traveling to many of its major cities. How is the matrix $A + B$ useful to you?

B. Mathematically, what is the difference between $(A + B)^2$ and $A^2 + B^2$? Be sure to show every step of your work.

C. Practically, what is the difference between $(A + B)^2$ and $A^2 + B^2$, to your trip?

4. After your trip to the DRP, you return to your job as an aerospace engineer. You are working on a spacecraft that, at a crucial point in its maneuvers, rotates in two stages. First the spacecraft rotates about the z-axis of space through an angle α. Three minutes later it rotates about the y-axis of space through an angle of β. Find a matrix that expresses the combined effect of the z-axis rotation followed by the y-axis rotation.

5. What's a *linear transformation*?

6. Each part A-H is a true/false question, but there are three valid answers: TRUE, FALSE, and PUNT. If you answer PUNT, then you receive half credit. Otherwise, if you answer correctly then you receive full credit, and if you answer incorrectly then you receive no credit. No explanation is necessary. Do not just write T, F, or P; write the entire word, and box your answer. Here, f and g denote linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, and A, B, and C denote $n \times n$ matrices.

A. Any $\vec{x}$ in $\ker f$ is also in $\ker(g \circ f)$.

B. Any $\vec{x}$ in $\ker(g \circ f)$ is also in $\ker f$.

C. There exists a 2×3 matrix with rank 3.

D. There exists a 3×2 matrix with rank 3.

E. If $A^2 = 0$, then $A = 0$ (where 0 means the 0 matrix).

F. If $AB = AC$, then $B = C$.

G. A and $\text{rref}(A)$ have the same kernel.

H. A and $\text{rref}(A)$ have the same image.