This exam begins for you when you open (or peek inside) this packet. It ends at 5:00 PM on Tuesday 2008 May 27. Between those two times, you may work on it as much as you like. I recommend that you get started early and work often. The exam is open-book and open-note, which means, precisely:

- You may freely consult all of this course's material: the Bretscher textbook, your class notes, your old homework and exam, and the materials on the course web site. If you missed a class and need to copy someone else's notes, do so before either of you begins the exam.
- You may assume all theorems discussed in class or in the assigned sections of the book. You do not have to prove or reprove them on this exam. On the other hand, you may not cite theorems that we have not studied. If you are unsure of whether you are allowed to cite a theorem, just ask.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may use a computer for these four purposes: viewing the course web site materials, running *Mathematica*, typing up your answers, and e-mailing with me. If you use *Mathematica*, then you may not load any packages other than *Combinatorica* (which is loaded in the 232.graphs.nb notebook on the course web site). You may use a hand-held calculator instead of *Mathematica*, if you like. You may not share any of these materials with another student.
- You may not discuss the exam in any way (spoken, written, pantomime, semaphore, etc.) with anyone but me until everyone has handed in the exam — even if *you* finish earlier. During the exam you will inevitably see your classmates around campus. Please refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.

During the exam you may want to ask me questions. You may ask clarifying questions for free. If you believe that the statement of a problem is wrong, then you should certainly ask for clarification. You may also ask for hints, which cost you some points, to be decided by me as I grade your paper. I will not give you a hint unless you unambiguously request it. I will try to check my e-mail over the weekend, but there is always some lag, and doing math over e-mail is not easy.

Your solutions should be thorough, self-explanatory, and polished (concise, neat, and well-written, employing complete sentences with punctuation). Always show enough work so that a classmate could follow your solutions. Do not show scratch work, false starts, circuitous reasoning, etc. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Submit your solutions in a single stapled packet, presented in the order they were assigned.

Partial credit is often awarded. Exam grades are loosely curved — by this I do not mean that there are predetermined numbers of As, Bs, Cs to be awarded, but rather that there are no predetermined scores required for grades A, B, C.

Good luck!

Orthogonal Complements

In $\mathbb{R}^4$, let

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} -1 \\ -2 \\ 4 \\ 2 \end{bmatrix}.$$

1. Find a basis for the orthogonal complement of the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.

Skew-Symmetric Matrices

Let A be any $n \times n$ skew-symmetric matrix.

2A. Show that the kth power of A is symmetric if k is even and skew-symmetric if k is odd.

2B. Show that if n is odd then A cannot be invertible.

Matrices For Inner Products

In class we've discussed how an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ can be constructed using an $n \times n$ matrix A, by the formula

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^\top A \, \vec{w}.$$

However, we were vague about exactly which matrices could be used for this purpose. An $n \times n$ matrix A is said to be *positive-definite* if for every nonzero vector $\vec{v} \in \mathbb{R}^n$,

$$\vec{v}^\top A \, \vec{v} > 0.$$

As you can check, A defines an inner product on $\mathbb{R}^n$ (by $\langle \vec{v}, \vec{w} \rangle = \vec{v}^\top A \, \vec{w}$) if and only if A is symmetric and positive-definite.

Recall that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if it preserves the dot product:

$$T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}.$$

In class we showed that this is equivalent to requiring that that matrix $[T]_\mathcal{E}$ of the linear transformation with respect to the standard basis $\mathcal{E}$ satisfy $[T]_\mathcal{E}^\top [T]_\mathcal{E} = I$. More generally, a linear transformation T is said to be *orthogonal* with respect to an inner product $\langle \cdot, \cdot \rangle$ if it preserves that inner product:

$$\langle T(\vec{v}), T(\vec{w}) \rangle = \langle \vec{v}, \vec{w} \rangle.$$

3A. Let $\langle \cdot, \cdot \rangle$ be an inner product defined by a symmetric, positive-definite matrix A as above. Find a condition (like $[T]_\mathcal{E}^\top [T]_\mathcal{E} = I$ — but it will be different from this) on the matrix $[T]_\mathcal{E}$ that determines whether or not a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$.

So far in this problem we've been using the standard basis $\mathcal{E}$. When A defines an inner product $\langle \cdot, \cdot \rangle$ with respect to $\mathcal{E}$, it makes sense to denote A by $[\langle \cdot, \cdot \rangle]_\mathcal{E}$, and to write the equation

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^\top A \, \vec{w}$$

as

$$\langle \vec{v}, \vec{w} \rangle = [\vec{v}]_{\mathcal{E}}^{\top} \, [\langle \cdot, \cdot \rangle]_{\mathcal{E}} \, [\vec{w}]_{\mathcal{E}}.$$

Let $\mathcal{B}$ be some other basis for $\mathbb{R}^n$. Then $\langle \cdot, \cdot \rangle$ should be represented by some matrix $[\langle \cdot, \cdot \rangle]_{\mathcal{B}}$ with respect to that basis, such that

$$\langle \vec{v}, \vec{w} \rangle = [\vec{v}]_{\mathcal{B}}^{\top} \, [\langle \cdot, \cdot \rangle]_{\mathcal{B}} \, [\vec{w}]_{\mathcal{B}}.$$

3B. Find a change-of-basis formula that relates $[\langle \cdot, \cdot \rangle]_{\mathcal{B}}$ to $[\langle \cdot, \cdot \rangle]_{\mathcal{E}}$.

Some Minus-Signs

For any vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$, define

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^{\top} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{w}.$$

This is similar to the dot product, but with a minus sign. Define $||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, as one usually defines a norm from an inner product. Unfortunately, this $\langle \cdot, \cdot \rangle$ doesn't define an inner product on $\mathbb{R}^2$, and this $|| \cdot ||$ doesn't define a norm, either.

4A. Which parts of the definition of inner product does $\langle \cdot, \cdot \rangle$ satisfy, and which parts does it not satisfy?

4B. For which vectors $\vec{v} \in \mathbb{R}^2$ is $||\vec{v}||$ defined? For which $\vec{v}$ is it 0? For which $\vec{v}$ is it 1? Answer these questions both in words/equations and in a detailed sketch of $\mathbb{R}^2$.

In class and homework (e.g. 2.2 #24) we have seen that any 2×2 special orthogonal matrix A can be written as

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some angle θ. That is, a determinant-1 transformation of $\mathbb{R}^2$ that preserves the dot product must be rotation by some angle θ.

Now I would like to develop a similar result for the $\langle \cdot, \cdot \rangle$ of this problem. Even though $\langle \cdot, \cdot \rangle$ is not an inner product, we can still say that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is *orthogonal* with respect to $\langle \cdot, \cdot \rangle$ if it preserves $\langle \cdot, \cdot \rangle$:

$$\langle T(\vec{v}), T(\vec{w}) \rangle = \langle \vec{v}, \vec{w} \rangle.$$

We can define a "rotation" to be a linear transformation with determinant 1 that is orthogonal with respect to $\langle \cdot, \cdot \rangle$. In order to characterize these $\langle \cdot, \cdot \rangle$-rotations, it is helpful to know the *hyperbolic trig functions*

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions obey many identities that are similar to the regular trig identities, but with slight differences in sign. For example, while $\cos^2 x + \sin^2 x = 1$, the hyperbolic trig functions instead satisfy

$$\cosh^2 x - \sinh^2 x = 1.$$

Here are some other facts [added to the exam in a revision]:

- $\cosh(-x) = \cosh x$.

- $\sinh(-x) = -\sinh x$.
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
- $\sinh(x+y) = \cosh x \sinh y + \sinh x \cosh y$.
- sinh is a bijection (a one-to-one, onto function) from $\mathbb{R}$ to $\mathbb{R}$, so there is an inverse function

$$\operatorname{arcsinh} : \mathbb{R} \to \mathbb{R}.$$

4C. Describe the $\langle \cdot, \cdot \rangle$-rotation matrices in terms of cosh **and** sinh**.**

Plumbing

The map below shows a rural area divided into several plots of land, with one house per plot. Currently the houses are not connected to any water/sewer system, but the landowners have negotiated with a nearby town to connect to its system at point P. The town is willing to pay for running a water main (a large pipe) due north from P and then diagonally across the landscape in a straight line. Each landowner must pay for running a small pipe from his/her house to the water main. Because the plots are narrow and the landowners don't want each others' pipes running across their land, they agree that all of these small pipes will run due north/south from the houses to the water main. Here is a possible solution:

[Picture omitted.]

The landowners have hired you as a consultant, to help them plan where the water main should go. You make a detailed survey of the area and determine that the houses are at the following coordinates, relative to P.

[Picture omitted. The coordinates of the houses are $(2, 5)$, $(4, 7)$, $(5, 6)$, $(6, 2)$, $(7, 8)$, $(8, 6)$.]

5. Using techniques from this course, find the route for the water main that minimizes the total length of small pipe that the landowners must use, in a least-squares sense.

Epilogue: The water main ended up running through a mountain lion habitat, an ancient burial ground, and an active volcano. There were no survivors.

Spacecraft Maneuvers

Recall from the previous exam that you are an aerospace engineer. A spacecraft traveling through space can rotate in three distinct ways: *pitch* (raising or lowering its nose), *yaw* (turning its nose left or right), and *roll* (turning about its front-back axis).

For example, if we place the spacecraft upright and pointing along the y-axis, then pitch is rotation about the x-axis, yaw is rotation about the z-axis, and roll is rotation about the y-axis. (All rotations are right-handed.) On the other hand, if we place the spacecraft upright and pointing along the x-axis, then pitch is rotation about the $-y$-axis, yaw is rotation about the z-axis, and roll is rotation about the x-axis.

[Picture omitted.]

6A. Start with a spacecraft upright and pointing along the y-axis. Draw what the spacecraft looks like after it yaws $\pi/2$ and then pitches $\pi/2$. In a separate picture, draw what the spacecraft looks like after it pitches $\pi/2$ and then yaws $\pi/2$.

6B. Again start with a spacecraft upright and pointing along the y-axis. The spacecraft is going to yaw θ and then pitch ϕ. Find a matrix that expresses the yaw. Find a matrix that expresses the pitch from the yawed position. Find a matrix that expresses the net effect of the yaw followed by the pitch.