1. Differentiate the following functions.

A. $y = f(x) = 5x^6 - 2x^3 + 3x + 1$.

Answer: $dy/dx = 30x^5 - 6x^2 + 3$.

B. $y = f(x) = 4^{\sin(\log(\sqrt{x}))}$.

Answer:

$$\begin{aligned} dy/dx &= 4^{\sin(\log(x^{1/2}))} \log(4) \cdot \frac{d}{dx} \sin(\log(x^{1/2})) \\ &= 4^{\sin(\log(x^{1/2}))} \log(4) \cdot \cos(\log(x^{1/2})) \cdot \frac{1}{x^{1/2}} \cdot \frac{1}{2} x^{-1/2} \\ &= 4^{\sin(\log(x^{1/2}))} \cdot \cos(\log(x^{1/2})) \cdot \frac{\log(4)}{2x}. \end{aligned}$$

2. We have seen a number of "limit laws" that help us manipulate limits. Is the following equation a valid limit law? That is, does it work for all functions f and g and all constants a?

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Answer: No, this is not always true. For example, let $a = 0$,

$$f(y) = \begin{cases} 0 & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases},$$

and $g(x) = x$. Then

$$f\left(\lim_{x \to 0} g(x)\right) = f(0) = 1,$$

but

$$\lim_{x \to 0} f(g(x)) = \lim_{x \to 0} f(x) = 0.$$

The problem here is that f is not continuous. The proposed limit law is, in fact, the very essence of continuity; it holds if and only if f is continuous at whatever $\lim_{x \to a} g(x)$ is. (See page 125.)

3. The table below shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from the earth to the sun) and their periods T (time of revolution in years). We wish to model T as a function of d.

Planet	d	T	Planet	d	T
Mercury	0.387	0.241	Jupiter	5.203	11.861
Venus	0.723	0.615	Saturn	9.541	29.457
Earth	1.000	1.000	Uranus	19.190	84.008
Mars	1.523	1.881	Neptune	30.086	164.784

A. When I graph the data on a log-log plot, they seem to lie on a line of slope 1.499 and intercept 0.000431. What, then, is the function $T(d)$?

Answer: The line on the log-log plot tells us that

$$\log T = 1.499(\log d) + 0.000431.$$

Therefore

$$T = e^{1.499(\log d)+0.000431} = e^{0.000431}\left(e^{\log d}\right)^{1.499} = e^{0.000431}d^{1.499}.$$

B. Kepler's Third Law of Planetary Motion says that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun." Does your model corroborate Kepler's Third Law?

Answer: Yes. Kepler says that $T^2 = kd^3$ for some constant k. In other words, $T = \sqrt{k}d^{1.5}$. This is very close to our answer, with $e^{0.000431}$ playing the role of $\sqrt{k}$.

(By the way, this is an almost-verbatim copy of Section 1.3 #26, which was assigned as homework.)

4. You are standing on a cliff above a river. 10 meters above you is a bridge across the river. A bungie jumper is attached to the bridge by a bungie (a large rubber band). She jumps off the bridge and then bounces up and down for a while on the bungie. She finds this entertaining. Let time $t = 0$ be the moment she jumps; her altitude, relative to yours, is thereafter given by

$$y(t) = 10e^{-t}\cos(1.5t)$$

(where altitude y is in meters and time t is in seconds).

A. Find a formula for her velocity. What are its units?

Answer: The units are m/s.

$$\begin{aligned} y'(t) &= -10e^{-t}\cos(1.5t) - 10e^{-t}\sin(1.5t)\cdot 1.5 \\ &= -10e^{-t}\left(\cos(1.5t) + 1.5\sin(1.5t)\right). \end{aligned}$$

B. Find a formula for her acceleration. What are its units?

Answer: The units are m/s^2.

$$\begin{aligned} y''(t) &= 10e^{-t}\left(\cos(1.5t) + 1.5\sin(1.5t)\right) - 10e^{-t}\left(-\sin(1.5t)\cdot 1.5 + 1.5\cos(1.5t)\cdot 1.5\right) \\ &= 10e^{-t}\left(\cos(1.5t) - 2.25\cos(1.5t) + 1.5\sin(1.5t) + 1.5\sin(1.5t)\right) \\ &= 10e^{-t}\left(3\sin(1.5t) - 1.25\cos(1.5t)\right). \end{aligned}$$

C. Roughly how much time does she take to bounce up and down once?

Answer: The bouncing arises from the $\cos(1.5t)$ part of $y(t)$. Cosine usually has a period of 2π, but in comparison $\cos(1.5t)$ has been compressed by a factor of 1.5 horizontally, so it has a period of $2\pi/1.5 = 4\pi/3$ (seconds). (Multiplying by e^{-t} slightly alters the distances between the peaks of the function, but only slightly. This $4\pi/3$ is fine as a rough answer.)

D. At what altitude, relative to yours, does she eventually come to rest?

Answer: The question asks what happens to y as t becomes large. Well, e^{-t} goes to 0, and the rest of the function just oscillates between -10 and 10. So y goes to 0. That is, the jumper ends up at my altitude.

5. Differentiate $y = \sin(2x)$ from the definition of the derivative. In the course of doing so, you may assume the following two limits; any other limits you use must be explained thoroughly.

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

Answer:

$$\begin{aligned}
\frac{d}{dx}\sin(2x) &= \lim_{h\to 0} \frac{\sin(2(x+h)) - \sin(2x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(2x+2h)) - \sin(2x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(2x)\cos(2h) + \cos(2x)\sin(2h) - \sin(2x)}{h} \\
&= \lim_{h\to 0} \sin(2x)\frac{\cos(2h)-1}{h} + \cos(2x)\frac{\sin(2h)}{h} \\
&= \sin(2x)\left(\lim_{h\to 0}\frac{\cos(2h)-1}{h}\right) + \cos(2x)\left(\lim_{h\to 0}\frac{\sin(2h)}{h}\right) \\
&= \sin(2x)\cdot 2\left(\lim_{h\to 0}\frac{\cos(2h)-1}{2h}\right) + \cos(2x)\cdot 2\left(\lim_{h\to 0}\frac{\sin(2h)}{2h}\right) \\
&= \sin(2x)\cdot 2\left(\lim_{u\to 0}\frac{\cos(u)-1}{u}\right) + \cos(2x)\cdot 2\left(\lim_{u\to 0}\frac{\sin(u)}{u}\right),
\end{aligned}$$

where $u = 2h$, so that $u \to 0$ as $h \to 0$. Then, using the given limits (with u substituted for h) the derivative becomes

$$\sin(2x)\cdot 2\cdot 0 + \cos(2x)\cdot 2\cdot 1 = \cos(2x)\cdot 2.$$

(You can check the answer by computing the derivative using the chain rule.)

6. You are trying to make money on the stock market. You select a particular stock, download data about how its price has changed over time, and run the data through a statistics program to create a model that, you hope, can predict the future. The program says that the data are described well by

$$p(t) = 10 + \frac{t}{(t-1)^2(t-2)(t-3)^2},$$

where t is time (in years from now) and p is the stock price (in U.S. dollars).

A. What is the practical meaning of $p'(5)$? What are its units? (Do not compute it.)

Answer: It is the instantaneous rate of change of the stock price five years from now, in dollars per year. In other words, it is a prediction of how fast the price will be rising, five years from now.

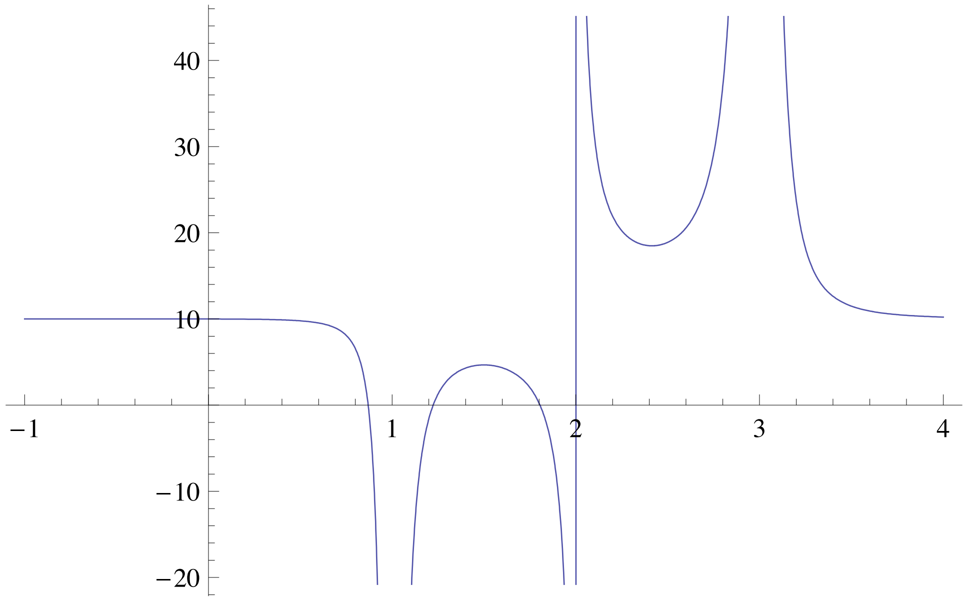

Figure 1: A more precise graph of the model $p(t)$ than that crafted in #6B. There should be no vertical line at $t = 2$; that should be an asymptote.

B. In order to make a lot of money, you would like to purchase the stock when its price is low and later sell the stock when its price is high. According to the model $p(t)$ given, when are the best times for you to buy and sell?

Answer: One way to proceed is to sketch a graph of $p(t)$. It has asymptotes at $t = 1, 2, 3$. If t is just to the left of 1, then

$$p(t) = 10 + \frac{t}{(t-1)^2(t-2)(t-3)^2} \approx 10 + \frac{1}{\epsilon(-1)(4)},$$

where ϵ is a small positive number; therefore the fraction is a large negative number. Thus we see that

$$\lim_{t\to 1^-} p(t) = -\infty.$$

By similar reasoning, we can work out that

$$\lim_{t\to 1^+} p(t) = \lim_{t\to 2^-} p(t) = -\infty,$$

and

$$\lim_{t\to 2^+} p(t) = \lim_{t\to 3^-} p(t) = \lim_{t\to 3^+} p(t) = \infty.$$

This allows us to make a rough sketch of the graph (a precise graph appears in Figure 1). Based on this model, we could buy just before $t = 2$ and sell right after $t = 2$ to make a tremendous profit.

C. Do you think that the $p(t)$ that your program found is a good model?

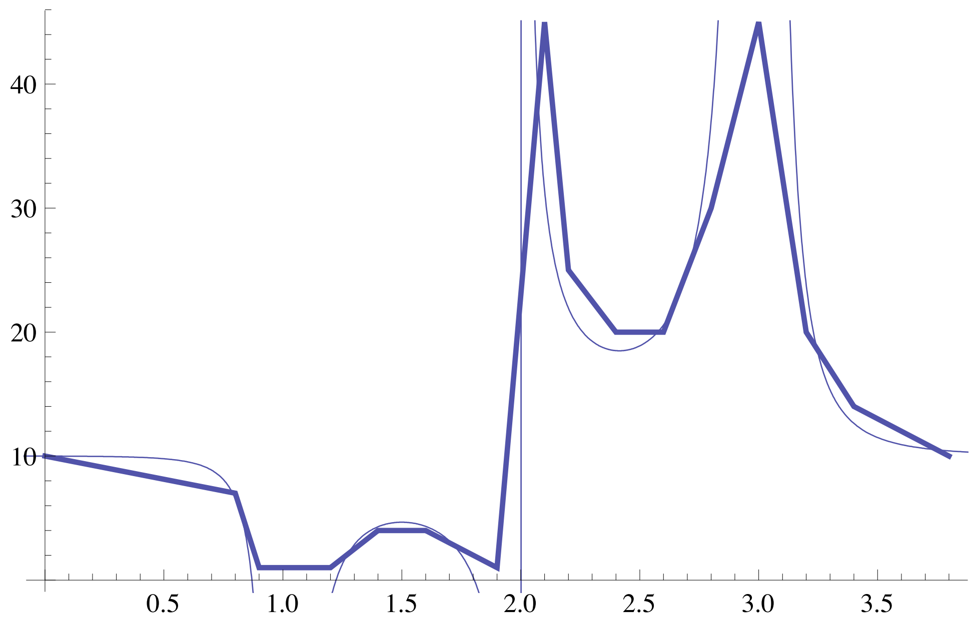

Figure 2: Although its asymptotes and negative values make $p(t)$ unrealistic, it is fairly close to this plot of (made-up) stock data that is realistic. See #6C.

Answer: It's not very good. For one thing, it sometimes predicts negative values, such as near $t = 1$, which is unrealistic because stock prices can never be negative. Furthermore, it has asymptotes, where it is undefined. Real stock prices do not. On the other hand, stock prices really do exhibit erratic behavior which, while not truly asymptotic, may be approximated fairly well by asymptotes (see Figure 2). Near the asymptotes, the model varies so wildly (meaning that it is very steep — small changes in time produce large changes in stock price) that I would be reluctant to bet money on it; but if the stock also varies wildly at those times, then at least the model has warned me to be careful buying or selling then.

7. The sun is a big ball of gas not far from where you're sitting. For the sake of this problem, we will assume that it has uniform density (which therefore equals its mass divided by its volume). The sun's radius is $6.96 \cdot 10^8$ meters and its density is 1.41 tonnes per cubic meter. It loses $1.26 \cdot 10^{14}$ tonnes of mass per year. I want to figure out how fast its radius is shrinking.

A. First, identify all of the relevant quantities in the problem, and indicate how they relate to each other — that is, which quantities depend on which other ones?

Answer: The mass m depends on the volume V by $m = 1.41V$. The volume V depends on the radius r by $V = \frac{4}{3}\pi r^3$. The radius r depends on time t in some unknown way.

B. Solve the problem — that is, compute how fast the radius is shrinking.

Answer: By the chain rule,

$$\frac{dm}{dt} = \frac{dm}{dr} \cdot \frac{dr}{dt}.$$

Now

$$m = 1.41V = 1.41\frac{4}{3}\pi r^3,$$

so

$$\frac{dm}{dr} = 1.41 \cdot 4\pi r^2,$$

which is $1.41 \cdot 4\pi \left(6.96 \cdot 10^8\right)^2 = 1.41 \cdot 4\pi \cdot 6.96^2 \cdot 10^{16}$ right now. So

$$\frac{dr}{dt} = \frac{dm/dt}{dm/dr} = \frac{-1.26 \cdot 10^{14}}{1.41 \cdot 4\pi \cdot 6.96^2 \cdot 10^{16}} = \frac{-1.26}{1.41 \cdot 4\pi \cdot 6.96^2 \cdot 10^2}.$$

By the way, this is a loss of about 0.000015 meters per year.

8. Prove that for any real number n the derivative of $y = x^n$ is $dy/dx = nx^{n-1}$. (Hint: Use logarithmic differentiation.)

Answer: If $y = x^n$ then $\log y = \log(x^n) = n \log x$. Implicit differentiation yields

$$\frac{1}{y}\frac{dy}{dx} = n\frac{1}{x},$$

which implies that

$$\frac{dy}{dx} = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.$$

9. Suppose that an environment of constant ambient temperature A contains a body of varying temperature $y = y(t)$. Then we have seen that

$$\frac{dy}{dt} = k(A - y),$$

where k is a positive constant. (You do not have to solve this differential equation here.)

A. If y starts out greater than A, then is dy/dt positive or negative? So what happens to y, according to the differential equation?

Answer: If $y > A$ then $A - y < 0$ so $dy/dt < 0$, which means that y decreases.

B. Similarly, if y starts out less than A, then what happens to it?

Answer: If $y < A$ then $A - y > 0$ so $dy/dt > 0$, which means that y increases.

C. Similarly, if y starts out equal to A, then what happens to it?

Answer: If $y = A$ then $dy/dt = 0$, so there is no change in y. In fact, the body will remain at temperature A forever.