

1. Show, using any method you like, that $\int \log x \, dx = x \log x - x + C$.

Answer: $\frac{d}{dx}(x \log x - x + C) = x \cdot \frac{1}{x} + 1 \cdot \log x - 1 + 0 = \log x$. Thus $\int \log x \, dx = x \log x - x + C$.

2. Compute these. Remember to put boxes around your answers.

A. $\int \sqrt{t} \, dt$

Answer: $\int t^{1/2} \, dt = \frac{2}{3}t^{3/2} + C$.

B. $\int \sin(2\pi\theta) \, d\theta$

Answer: $\int \sin(2\pi\theta) \, d\theta = -\frac{1}{2\pi} \cos(2\pi\theta) + C$.

C. $\int x^2 \log(x^3) \, dx$

Answer: Let $u = x^3$, so $\frac{du}{dx} = 3x^2$ and $\frac{1}{3}du = x^2 \, dx$. Then

$$\int x^2 \log(x^3) \, dx = \int (\log u) \frac{1}{3} \, du = \frac{1}{3} \int \log u \, du = \frac{1}{3}(u \log u - u + C) = \frac{1}{3}(x^3 \log x^3 - x^3 + C).$$

Here we have used the result of Problem 1.

3. Suppose that $y = f(x)$ is a continuous function and a and b any two numbers. Thoroughly discuss the distinction between these two objects:

$$\int f(x) \, dx \quad \text{and} \quad \int_a^b f(x) \, dx.$$

Answer: $\int f(x) \, dx$ is the antiderivative (or indefinite integral) of $f(x)$. It is a function, or rather a family of functions, namely all of the functions whose derivative is $f(x)$. If $F(x)$ is any one of these functions, then all of the others are of the form $F(x) + C$ for some constant C . In contrast, $\int_a^b f(x) \, dx$ is the (definite) integral of $f(x)$ over the interval $[a, b]$. It is a number, representing the (signed) area of the region between the graph of $y = f(x)$ and the x -axis. It is defined as a limit of sums of areas of rectangles that, taken together, approximate that region.

4. Crows like to eat various mollusks. To crack open these hard-shelled creatures, they take them up in the air and drop them onto rocks. The longer the drop, the harder the impact and the more likely the mollusk is to crack open. On the other hand, flying to high altitudes requires a lot of energy. When trying to crack a mollusk, the crow naturally selects its dropping altitude in a way that minimizes the total energy required.

Let h be the crow's dropping altitude (in m). Assuming that the crow and mollusk together have mass 1kg, then each flight to altitude h requires the crow to expend energy h (in kg m²/s²). Also, it turns out that the number of drops required (on average) is

$$n(h) = 1 + \frac{16}{h - 1.2}.$$

A. Write a function $f(h)$ that describes the total energy required to crack a mollusk.

Answer: The energy per flight is h , and the number of flights required is $n(h)$, so the total energy required is

$$f(h) = h \cdot n(h) = h \left(1 + \frac{16}{h - 1.2} \right) = \frac{h(h - 1.2) + 16h}{h - 1.2} = \frac{h^2 + 14.8h}{h - 1.2}.$$

It is worth noting that $f(h)$ is undefined at $h = 1.2$ and negative for $0 < h < 1.2$. Therefore the only sensible interval for h is $(1.2, \infty)$.

B. What value of h should the crow choose, to minimize its total energy expenditure?

Answer: The derivative is [omitting some work]

$$f'(h) = \dots = \frac{h^2 - 2.4h - (1.2)(14.8)}{(h - 1.2)^2}.$$

For h in $(1.2, \infty)$ this is always defined. It is zero when $h^2 - 2.4h - (1.2)(14.8) = 0$; that is, when

$$\begin{aligned} h &= \frac{2.4 \pm \sqrt{(2.4)^2 + 4(1.2)(14.8)}}{2} \\ &= 1.2 \pm \sqrt{(1.2)^2 + (1.2)(14.8)} \\ &= 1.2 \pm 4\sqrt{1.2}. \end{aligned}$$

The solution $1.2 - 4\sqrt{1.2}$ is outside $(1.2, \infty)$, so $h = 1.2 + 4\sqrt{1.2}$. This is the only critical point of $f(h)$ on $(1.2, \infty)$. There are many ways to see that it is a local minimum. For one, $h = 1.2 + 4\sqrt{1.2}$ is the right-hand root of the parabola $h^2 - 2.4h - (1.2)(14.8) = 0$, which opens up. Thus $h^2 - 2.4h - (1.2)(14.8) = 0$ passes from negative to positive at $h = 1.2 + 4\sqrt{1.2}$, and so must $f'(h)$, since its denominator is positive near $h = 1.2 + 4\sqrt{1.2}$. Since $f'(h)$ passes from negative to positive, $h = 1.2 + 4\sqrt{1.2}$ is a local minimum. Then, because it is the only critical point on $(1.2, \infty)$, it must also be the absolute minimum on that interval.

Thus the best dropping altitude is $h = 1.2 + 4\sqrt{1.2}$ m. The average number of drops there is

$$n(1.2 + 4\sqrt{1.2}) = 1 + \frac{4}{\sqrt{1.2}},$$

and the total energy spent is

$$f(1.2 + 4\sqrt{1.2}) = (1.2 + 4\sqrt{1.2}) \left(1 + \frac{4}{\sqrt{1.2}} \right) \text{ kg m}^2/\text{s}^2.$$

[By the way, these work out to be $h \approx 5.6$ m, $n(h) \approx 4.6$, and $f(h) \approx 26$ kg m²/s². These results are corroborated by actual observations of crow behavior.]

5. Recall that the erf function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

A. What is the domain of erf? That is, which values of x can be plugged into it?

Answer: Any real number can be plugged in for x .

B. It turns out that e^{-t^2} is difficult to antidifferentiate, so to compute erf we must resort to numerical approximations. In as much detail as possible, explain/show how one could estimate $\text{erf}(x)$ for any given number x using Riemann sums (sums of areas of rectangles).

Answer: Let n denote a positive integer, $\Delta t = \frac{x-0}{n} = \frac{x}{n}$, and $t_k = 0 + k\Delta t = k\frac{x}{n}$. Then

$$\begin{aligned}\text{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \\ &= \frac{2}{\sqrt{\pi}} \lim_{n \rightarrow \infty} \sum_{k=1}^n e^{-t_k^2} \Delta t \\ &= \lim_{n \rightarrow \infty} \frac{2}{\sqrt{\pi}} \sum_{k=1}^n e^{-(k\frac{x}{n})^2} \frac{x}{n}.\end{aligned}$$

To approximate $\text{erf}(x)$, we simply pick a large value of n and compute

$$\frac{2}{\sqrt{\pi}} \sum_{k=1}^n e^{-(k\frac{x}{n})^2} \frac{x}{n}.$$

6. Recall from the previous page that $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

A. What is $\frac{d}{dx}\text{erf}(x)$?

Answer: By the fundamental theorem of calculus,

$$\frac{d}{dx}\text{erf}(x) = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} e^{-x^2}.$$

B. I want to solve $\text{erf}(x) = 0.25$, but this is difficult, so I resign myself to an approximate solution. In as much detail as possible, explain/show how one could find an approximate solution using Newton's method.

Answer: Let $f(x) = \text{erf}(x) - 0.25$; then solving $\text{erf}(x) = 0.25$ is tantamount to finding a zero of $f(x)$. Notice that $f'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$ by Part A of this problem. Newton's method uses the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\text{erf}(x_n) - 0.25}{\frac{2}{\sqrt{\pi}}e^{-x_n^2}}.$$

Notice that this formula requires us to compute $\text{erf}(x_n)$, which is difficult in itself; we have to use an approximation, as in Problem 5. Assuming that we have already overcome that obstacle, we pick a starting value x_1 and then compute x_2, x_3, x_4, \dots using the iteration formula, until we get answers that agree to the desired number of decimal places.

[By the way, solving $\text{erf}(x) = 0.25$ (or $\text{erf}(x) = a$ for some other a) is useful in the real world for computing percentiles in data that is normally distributed. For example, computations like this are carried out to determine percentiles for SAT scores.]

7. My dentist keeps asking me about the following function for some reason. To get her off my back, please make a detailed graph of it, including intercepts, asymptotes, critical points, inflection points, the correct increasing/decreasing and concavity behavior, local maxima and minima, and anything else that you think is significant.

$$f(x) = x^{8/3} - x^{2/3}.$$

Answer: First, it is worth noting that $f(x)$ is an even function, so its graph is symmetric with respect to the y -axis. Now $0 = f(x) = x^{8/3} - x^{2/3}$ if and only if $x = 0$ or $x^2 - 1 = 0$, so the zeros of f are $x = -1, 0, 1$. The derivative is

$$f'(x) = \frac{8}{3}x^{5/3} - \frac{2}{3}x^{-1/3}.$$

This is undefined at $x = 0$; it is zero if and only if

$$\begin{aligned} \frac{8}{3}x^2 - \frac{2}{3} &= 0 \\ \Leftrightarrow x^2 &= \frac{1}{4} \\ \Leftrightarrow x &= \pm \frac{1}{2}. \end{aligned}$$

So the critical points are $x = -\frac{1}{2}, 0, \frac{1}{2}$. Now let's analyze the increasing/decreasing behavior. For $x > 0$,

$$\begin{aligned} f'(x) = \frac{8}{3}x^{5/3} - \frac{2}{3}x^{-1/3} &> 0 \\ \Leftrightarrow \frac{8}{3}x^2 - \frac{2}{3} &> 0 \\ \Leftrightarrow x &> \frac{1}{2}. \end{aligned}$$

Hence $f(x)$ is increasing for $x > \frac{1}{2}$ and decreasing for $0 < x < \frac{1}{2}$. Because the graph is symmetric about the y -axis, $f(x)$ is decreasing for $x < -\frac{1}{2}$ and increasing for $-\frac{1}{2} < x < 0$. Finally, the second derivative is

$$f''(x) = \frac{40}{9}x^{2/3} + \frac{2}{9}x^{-4/3}.$$

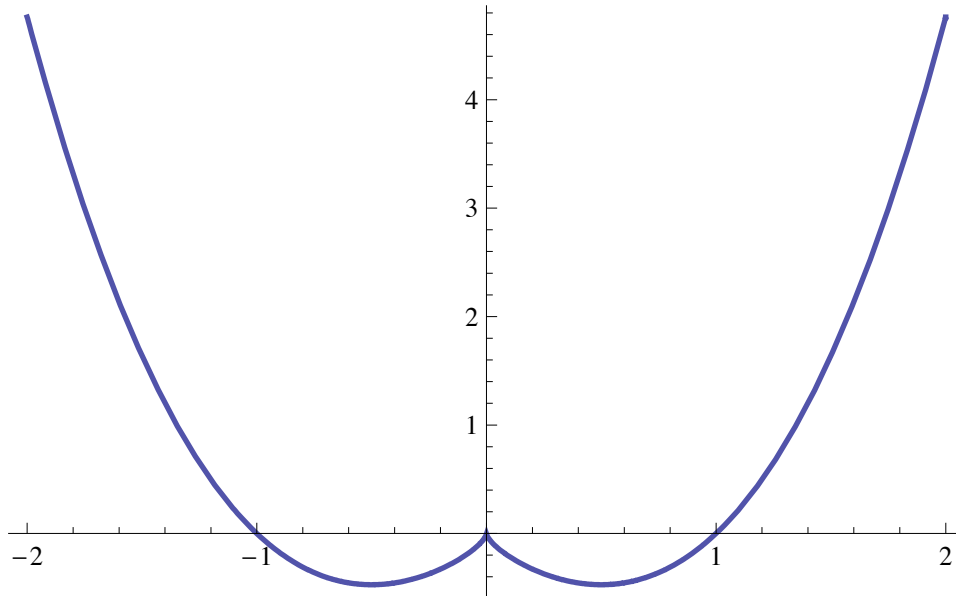
This is always positive, so the graph of $y = f(x)$ is always concave up and the critical points at $x = \pm\frac{1}{2}$ are local minima. We now know enough to make a good sketch [see Figure 1, which I've generated by computer, for the sake of this answer key].

8. A space probe is shooting away from Earth. Its velocity at hour t is $27000 + 0.1\sqrt[3]{t}$ km/hr.

A. Write an integral that represents how far the probe travels in the first 24 hours.

Answer: In kilometers, the distance traveled is

$$\int_0^{24} 27000 + 0.1\sqrt[3]{t} dt.$$

Figure 1: The graph of $y = x^{8/3} - x^{2/3}$ from Problem 7.

B. Compute the integral.

Answer:

$$\begin{aligned} \int_0^{24} 27000 + 0.1t^{1/3} dt &= \left[27000t + 0.1 \frac{3}{4} t^{4/3} \right]_0^{24} \\ &= \left(27000 \cdot 24 + 0.1 \frac{3}{4} \cdot 24^{4/3} \right) - \left(27000 \cdot 0 + 0.1 \frac{3}{4} \cdot 0^{4/3} \right) \\ &= 27000 \cdot 24 + 0.1 \frac{3}{4} \cdot 24^{4/3}. \end{aligned}$$

9. This problem concerns linear approximation.

A. Find the linear approximation to $y = \cos x$ at $x = \pi/3$.

Answer: First, $\frac{dy}{dx} = -\sin x$, so the slope at $x = \pi/3$ is $-\sin(\pi/3) = -\frac{\sqrt{3}}{2}$. Thus the tangent line is

$$\begin{aligned} y - \cos(\pi/3) &= \frac{-\sqrt{3}}{2}(x - \pi/3) \\ \Leftrightarrow y &= \frac{-\sqrt{3}}{2}x + \frac{\sqrt{3}\pi}{6} + \frac{1}{2}. \end{aligned}$$

B. Using your linear approximation from Part A, estimate $\cos(1)$. Your answer may involve things like $\sqrt{\quad}$ and π .

Answer: $\cos(1) \approx \frac{-\sqrt{3}}{2} + \frac{\sqrt{3}\pi}{6} + \frac{1}{2}$.

C. Using your linear approximation again, estimate $\cos(-1)$. Is this estimate good or not?

Answer: According to our linear approximation, $\cos(-1) \approx \frac{\sqrt{3}}{2} + \frac{\sqrt{3}\pi}{6} + \frac{1}{2}$. But this is probably not a good estimate, because -1 is far away from $\pi/3$, where we took the approximation.