You have 150 minutes.

You may not use any calculator. You may use one standard-size ($8.5 \times 11$ inches) sheet of paper with notes written by you on both sides.

Always show your work and explain all of your answers. Good work often earns partial credit. A correct answer with no explanation often earns little or no credit.

Whenever possible, give a simplified answer and put a box around your answer so that it can be found easily.

Hints will not be given out on this test. You should feel free to ask clarifying questions, however.

Good luck.

**1**. I want to compute $\int_2^7 \sin(\sqrt{x+1})\,dx$. In detail, set up this integral as a limit of Riemann sums — but do not evaluate it.

**2**. Differentiate. Do not simplify, but do box your answers.

A. $\dfrac{d}{dt}\cos 3t$

B. $\dfrac{d}{dx}4^{\sin\log\sqrt{x}}$

C. $\dfrac{d}{du}\left(3e^{2u}\right)\left(u^5-2u+1\right)$

D. $\dfrac{d}{dx}(\log x)^{x^2-x}$

**3**. Antidifferentiate. Simplify and box your answers.

A. $\displaystyle\int x^{5/7}\,dx$

B. $\displaystyle\int \frac{-5}{x} + 3x^{-2}\,dx$

C. $\displaystyle\int (\sin t)e^{1+\cos t}\,dt$

D. $\displaystyle\int \frac{\sqrt{4+\log u}}{u}\,du$

**4**. Let $y = f(x) = x^2$.

A. Give the Newton's method iteration formula for this function. Simplify.

B. Using a seed value of $x_1 = 1$, carry out several iterations until a pattern emerges. Discuss.

**5**. Let $y = f(x) = e^{-x}$.

A. Give the Newton's method iteration formula for this function. Simplify.

B. Using a seed value of $x_1 = 1$, carry out several iterations until a pattern emerges. Discuss.

**6**. The acclaimed actress Meryl Streep is trying to model the following data with $y$ as a function of $x$. On a semilog plot, they seem to lie along a line of slope $-0.33$ and intercept 2. What, then, is her model function $y(x)$?

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 7.38906 | 5.29449 | 3.79367 | 2.71828 | 1.94773 |

**7**. What is the general solution to the differential equation $\frac{dy}{dx} = ky$? No explanation needed.

**8**. Recall that Newton's law of cooling can be expressed as the differential equation

$$\frac{dy}{dt} = k(A - y).$$

Express/explain this differential equation in words. Do not use any symbols.

**9**. After many months, a probe traveling through space at a frigid $-240°$C finally reaches its destination — the thrilling and mysterious planet Venus! Upon entering the torrid (460°C) Venusian atmosphere, the probe immediately begins warming. After 1 hour its temperature is 110°C. When its temperature reaches 285°C, it will stop functioning. Use Newton's law of cooling/warming to determine how long the probe can function in the Venusian atmosphere.

**10**. Find the area enclosed by the graphs of $y = x^3 - 2x$ and $y = 7x$.

**11**. Sketch a single function $y = f(x)$, defined and continuous on $(-\infty, \infty)$, that satisfies all of these properties: $f'(x) < 0$ for $x < -1$, $f'(-1) = 0$, $f'(x) > 0$ for $-1 < x < 1$, $f'(1)$ is undefined, $f'(x) > 0$ for $x > 1$, $f''(x) > 0$ for $x < 0$, $f''(0) = 0$, and $f''(x) < 0$ for $x > 0$.

**12**. A historical linguist is studying the Sivad civilization, which thrived in northern Asia many centuries ago. The Sivad inhabited a large circular region, whose radius expanded at a rate of 3 km/yr (as estimated from archaeological artifacts). As their region grew in area, the number of words in their language grew (due to encounters with other cultures, fragmentation among the Sivad, etc.). Based on newly discovered written records, the linguist estimates that the language was growing at a rate of 50 words/yr when the Sivad covered a region of radius 200 km. At that time, how many words were being generated for each square kilometer of land?

**13**. Let $b$ be a positive number. Let $R$ be the region enclosed by $x = 0$, $x = b$, $y = 0$, and $y = e^{-x}$.

A. Compute the volume of the solid obtained by rotating $R$ about the $x$-axis. Your answer will be in terms of $b$.

B. What happens to the volume, as $b$ goes to infinity?

**14**. Let $y = f(x)$ be a differentiable function and $a$ any number. Suppose that we know how to compute $f(a)$. Then we can estimate $f(b)$ for any $b$ near $a$ by simply declaring that $f(b) \approx f(a)$. But I wonder: How far off is this estimate?

Prove that there is some $c$ between $a$ and $b$ such that the error in the estimate is $f'(c) \cdot (b - a)$.

**15**. Climate scientists in the Swiss Alps have been studying Musterhorn Glacier B, which I just made up. This is a giant, rectangular block of ice. Let $w$ denote its width, $h$ its height, and $\ell$ its length, all in kilometers. Its volume is $v = wh\ell$. Its height and length have been decreasing over recent years due to climate change, as the chart below shows. Its width is a constant 1.31 km, because it is trapped in a steep valley between two mountains.

| Year $t$ | Height $h(t)$ (in km) | Length $\ell(t)$ (in km) |
|---|---|---|
| 1980 | 0.82 | 2.15 |
| 1990 | 0.80 | 2.12 |
| 2000 | 0.79 | 2.10 |

A. Numerically estimate $h'(t)$ at $t = 1990$.

B. Numerically estimate $\ell'(t)$ at $t = 1990$.

C. Using your answers to Parts A and B, and the product rule, estimate the rate of change of the volume of the glacier in 1990.

**16**. The graph of $y = f(x)$ is given below. Let's define

$$g(x) = \int_0^x f(t)\, dt.$$

On the same plot, sketch the graph of $y = g(x)$.

**17**. Assuming the first fundamental theorem of calculus (the one with the $\frac{d}{dx}$), prove the second fundamental theorem (the one with the $F(b) - F(a)$).

**18**. You run a charity, and you are trying to get the public to donate money. You decide to place some advertisements on the web. The initial problem of locating a good site costs you $b$ dollars. You are then charged $a$ dollars per ad after you find the site. From past experience, you suspect that if you place $x$ ads, then you will receive $k\sqrt{x}$ dollars in donations. How many ads should you place, to make the most money for your charity? How much money do you make? (Your answers will be in terms of the constants $a$, $b$, and $k$.)