One-Page Summary of Algebra, by Joshua R. Davis, jdavis@math.wisc.edu, 31 Dec 2003

Precedence conventions determine the order in which operations are applied in complicated formulae. From first to last, the order is: parentheses, exponents, multiplication/division, addition/subtraction. For example, $3(4+1)^2 - 2$ means "add 4 and 1, square the result, multiply that by 3, and subtract 2".

There are nine basic rules of algebra, from which every other formula follows. (You should be able to prove everything that comes later from these!) First, for any numbers a, b, and c, addition satisfies:

1. Associativity: $a + (b + c) = (a + b) + c$.
2. Identity: $a + 0 = a$.
3. Inverses: $a + -a = 0$.
4. Commutativity: $a + b = b + a$.

Next, we have a similar set of rules for multiplication:

5. Associativity: $a(bc) = (ab)c$.
6. Identity: $a1 = a$.
7. Inverses: $aa^{-1} = 1$, except when $a = 0$; 0 does not have an inverse.
8. Commutativity: $ab = ba$.

Finally, we have a single rule that governs the interaction between addition and multiplication:

9. Distributivity: $a(b + c) = ab + ac$.

Where are subtraction and division? Well, $a - b$ is defined as $a + -b$, and a/b is defined as ab^{-1}. Notice that, because 0 does not have a multiplicative inverse, it is impossible to divide by 0; never, ever do it! From the definition of division follow these rules:

- $a^{-1} = 1/a$,
- $(a/b)(c/d) = ab^{-1}cd^{-1} = (ac)/(bd)$, and
- $(a/b)^{-1} = b/a$.

Fractions are a frequent source of errors. In order to add (or subtract) them, you need a common denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$$

You can cancel a factor from the top and bottom of a fraction, but only if it's in *every* term; for example,

$$\frac{ab - ac}{ad + ae} = \frac{a(b - c)}{a(d + e)} = \frac{a}{a} \cdot \frac{b - c}{d + e} = \frac{b - c}{d + e}.$$

Multiplying something by an integer is the same thing as adding it (or its negative) to itself some number of times. For example, $3a = a + a + a$, and $-3a = -a + -a + -a$. Similarly, integer powers represent repeated multiplication: $a^3 = aaa$, and $a^{-3} = a^{-1}a^{-1}a^{-1}$. For any $a, b \neq 0$ and any integers m, n,

- $a^m a^n = a^{m+n}$,
- $(a^m)^n = a^{mn}$, and
- $(ab)^n = a^n b^n$.

Roots are the "opposites" of powers. For example, $\sqrt[3]{8} = 2$, since $2^3 = 8$. In fact, we use fractional powers to denote roots: $a^{1/n}$ is the nth root, $\sqrt[n]{a}$. Then fractional exponents behave much like integer exponents: $a^{1/n}b^{1/n} = (ab)^{1/n}$, and $(a^{1/n})^n = a^{n/n} = a^1 = a$.

Unfortunately, the similar rule $(a^n)^{1/n} = a$ is not always true. For example, $((-3)^2)^{1/2} = 9^{1/2} = 3$, whereas we'd have liked to get -3. This problem arises because there is more than one candidate for what the square root should be. By convention, when we write $\sqrt{9}$, we mean the positive square root, 3, although -3 is an equally good answer. In short, $(a^n)^{1/n} = a$ for odd n, but $(a^n)^{1/n} = |a|$ for even n.

Here are some miscellaneous things to keep in mind:

- For any number a, $a0 = 0$. If $ab = 0$, then $a = 0$ or $b = 0$; if $a/b = 0$, then $a = 0$.
- $(a + b)(c + d) = ac + ad + bc + bd$; for example, $(a + b)^2 = a^2 + 2ab + b^2$, and $(a - b)(a + b) = a^2 - b^2$.
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
- Given numbers a, b, and c, the two solutions to the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- $(a + b)^2 \neq a^2 + b^2$, and $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$.
- $a - (b + c) = a - b - c \neq a - b + c$.