The following results on connected spaces were presented in class, but the proofs had unacceptable errors, so I am obliged to give you these corrected proofs.

Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and N is strictly between $f(a)$ and $f(b)$, then there exists some $c \in [a, b]$ such that $f(c) = N$.

Proof. Without loss of generality, suppose $f(a) \leq f(b)$, so that $(f(a), f(b))$ is an interval. Let $N \in (f(a), f(b))$; we wish to show that $N \in f([a, b])$. Let $U = (-\infty, N) \cap f([a, b])$ and $V = (N, \infty) \cap f([a, b])$. Then

- U and V are open in $f([a, b])$, since each is the intersection of $f([a, b])$ with an open set in $\mathbb{R}$.
- U and V are disjoint because $(-\infty, N)$ and (N, ∞) are disjoint.
- $f(a) \in (-\infty, N)$ and $f(a) \in f([a, b])$, so $f(a) \in U$ and U is nonempty. Similarly, $f(b) \in V$ so V is nonempty.
- $U \cup V = f([a, b]) \cap ((-\infty, N) \cup (N, \infty)) = f([a, b]) - \{N\}$.

In particular, if N is not an element of $f([a, b])$, then $U \cup V = f([a, b])$ and so $\{U, V\}$ constitutes a separation of $f([a, b])$. This is impossible, since $f([a, b])$, being the continuous image of the connected space $[a, b]$, is connected. Thus $N \in f([a, b])$.

Theorem: If $A \subset X$ is a connected subset (a connected space, in the subspace topology), then the closure $\bar{A}$ is also connected.

Proof. Let's prove the contrapositive. Suppose that $\bar{A} = U_1 \cup U_2$ for U_1, U_2 disjoint nonempty open subsets of $\bar{A}$. By the definition of the subspace topology, $U_i = \bar{A} \cap V_i$ for some open subsets V_1, V_2 of X. Let $W_i = A \cap V_i$. Notice that

$$W_i = A \cap V_i = A \cap \bar{A} \cap V_i = A \cap U_i.$$

I claim that $\{W_1, W_2\}$ constitutes a separation of A. To wit:

- Since $W_i = A \cap V_i$ and V_i is open in X, W_i is open in the subspace A of X.
- W_1 and W_2 cover A:
$$W_1 \cup W_2 = (A \cap U_1) \cup (A \cap U_2) = A \cap (U_1 \cup U_2) = A \cap \bar{A} = A.$$
- Since the U_i are disjoint, so are the $W_i = A \cap U_i$.
- Suppose for the sake of contradiction that $W_i = A \cap V_i$ is empty. Then A is a subset of $X - V_i$, which is closed in X. Since $\bar{A}$ is the intersection of all closed sets of X containing A, we have $\bar{A} \subseteq X - V_i$. But then $U_i = \bar{A} \cap V_i = \emptyset$, contradicting the definition of U_i. So W_i is nonempty.

Thus $\{W_1, W_2\}$ is a separation of A, so A is not connected, as desired.