1. Connectedness And Liftings

Part B of this problem is a claim that we used in our lifting arguments in class but did not prove. I'm asking you to prove it now.

A. Let X be a connected topological space. Let $\{U_1, \dots, U_n\}$ be an open cover of X (with the U_i are nonempty and distinct). Suppose that the set $\{U_1, \dots, U_n\}$ is partitioned into two nonempty disjoint sets $\{V_1, \dots, V_m\}$, $\{W_1, \dots, W_\ell\}$. Prove that there is some W_k that intersects the union $\bigcup_{j=1}^m V_j$.

Answer: Suppose not. Then $\bigcup W_i$ is disjoint from $\bigcup V_j$. So these two sets are disjoint, nonempty, and open, and together they cover X. But this is impossible, since X is connected.

B. Suppose that $p : E \to B$ is a covering space. Let X be compact and connected and $f : X \to B$ continuous. Show that $f(X) \subseteq B$ can be covered by finitely many evenly covered open sets $U_1, \dots, U_n \subseteq B$ such that, for all $k = 2, \dots, n$,

$$f(X) \bigcap U_k \bigcap \left(\bigcup_{i=1}^{k-1} U_i \right) \neq \emptyset.$$

Answer: Since every point in B has some evenly covered neighborhood, B can be covered by evenly covered open sets, and so can $f(X) \subseteq B$. But $f(X)$ is compact, since X is compact and f is continuous. So $f(X)$ can be covered by finitely many evenly covered open sets $U_1, \dots, U_n \subseteq B$. Then $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$ is a finite open cover of X; by discarding empty or repeated sets, we may assume that the $f^{-1}(U_i)$ are nonempty and distinct.

We will now reorder these sets $f^{-1}(U_i)$ so that each one intersects the union of the previous ones. Pick any $f^{-1}(U_i)$ to start; call it V_1. Then, by Part A, one of the other open sets $f^{-1}(U_i)$ intersects V_1; call it V_2. Then, again by Part A, there is some other $f^{-1}(U_i)$ that intersects $V_1 \bigcup V_2$; call it V_3. Continue in this manner until $V_1, \dots, V_n$ have been chosen.

Relabel the U_i so that $V_i = f^{-1}(U_i)$. Then, for any $k \geq 2$, $V_k = f^{-1}(U_k)$ has a nonempty intersection with the set

$$\bigcup_{i=1}^{k-1} V_i = \bigcup_{i=1}^{k-1} f^{-1}(U_i),$$

and so

$$f(X) \bigcap U_k \bigcap \left(\bigcup_{i=1}^{k-1} U_i \right) \neq \emptyset.$$

2. Covering Spaces

Recall that an *n-manifold* is a Hausdorff space in which each point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. (This is the definition we have always used in class; it appears to be slightly weaker than the definition given on page 225.) Let $p : E \to B$ be a covering space. For the following two questions, prove or give a counterexample.

A. If B is an n-manifold, then must E also be an n-manifold?

Answer: Yes, E must be an n-manifold. First let us check Hausdorffness. Let e_1, e_2 be two distinct points in E. Then either $p(e_1) = p(e_2)$ or not.

If $p(e_1) = p(e_2)$, then e_1 and e_2 are distinct preimages of a single point $b \in B$. Let U be an evenly covered neighborhood of b. Then e_1, e_2 lie in distinct slices V_1, V_2 of E over U. By the definition of a covering map, V_1 and V_2 are open in E and disjoint. So V_1 and V_2 are disjoint neighborhoods of e_1 and e_2.

If $p(e_1) \neq p(e_2)$, then $p(e_1)$ and $p(e_2)$ have disjoint neighborhoods W_1 and W_2, because B is Hausdorff. There are also evenly covered neigborhoods U_1, U_2 about $p(e_1), p(e_2)$ respectively. Then $U_1 \bigcap W_1$ and $U_2 \bigcap W_2$ are disjoint evenly covered neighborhoods of $p(e_1)$ and $p(e_2)$, and $p^{-1}(U_1 \bigcap W_1)$ and $p^{-1}(U_2 \bigcap W_2)$ are disjoint neighborhoods of e_1, e_2.

So E is Hausdorff. Now we check the condition that every point in E has a neighborhood homemorphic to an open set in $\mathbb{R}^n$. Let $e \in E$. Let U be an evenly covered neighborhood of $p(e)$ and W a neighborhood of $p(e)$ that is homeomorphic to an open set in $\mathbb{R}^n$. Then $U \bigcap W$ is an evenly covered neighborhood of $p(e)$ that is also homeomorphic to an open set in $\mathbb{R}^n$. Since p is a covering map, $p^{-1}(U \bigcap W)$ is a disjoint union of open sets $V_\alpha \subseteq E$, each of which is homeomorphic to $U \bigcap W$ and hence to an open set in $\mathbb{R}^n$. The original point e is an element of one of these V_α; that V_α is the desired neighborhood.

B. If B is simply connected and E is path-connected, then must E also be simply connected?

Answer: Yes, E must be simply connected. Let $e \in E$ and let $[f] \in \pi_1(E, e)$ be a loop class. Then $p \circ f$ is a loop in B. But B is simply connected, so there exists a path homotopy between $p \circ f$ and the constant loop at $p(e)$. By Lemma 54.2 the path homotopy lifts to one between f and the constant loop at e. (See also Theorem 54.3.) Thus f is nulhomotopic, $\pi_1(E, e)$ is trivial, and E is simply connected.

Remark: This is essentially a rephrasing of Theorem 54.6a, which is proved using the same argument.

3. Malicious Art Curation

A. In a cartoon like those above, show how to arrange the cable so that while both nails are in the wall the painting stays up, but if *either* nail comes out of the wall then the painting crashes to the floor. (The only freedom you have is in how the cable is arranged. You are not allowed to change how it is attached to the painting, you are not allowed to knock holes in the wall, etc.)

Answer: Starting at the painting, draw a cable that goes up between the nails, goes around the left nail, goes straight across the wall until it is below the right nail, goes around the right nail, goes diagonally across the wall until it is below the left nail, goes around the left nail, goes straight across the wall until it is above the right nail, goes around the right nail, and returns to the painting. In short, if a is a loop based at x (the attachment point on the painting) that goes once around the left nail counterclockwise, and b is a loop based at x that goes once around the right nail counterclockwise, then the loop I'm going for is path-homotopic to $ab\bar{a}\bar{b}$.

Remark: This is not the only answer.

B. What does this have to do with our course? Equate the wall with $\mathbb{R}^2$, if you like.

Answer: The wall is $\mathbb{R}^2$. Let $p_1 \in \mathbb{R}^2$ be the location of the left-hand nail and $P_1 = \mathbb{R}^2 - \{p_1\}$ the rest of the wall. Define p_2, P_2 similarly for the other nail. Let $i_1 : P_1 \bigcap P_2 \hookrightarrow P_1$ and $i_2 : P_1 \bigcap P_2 \hookrightarrow P_2$ be the inclusion maps. When the painting is hanging, the cable forms a loop in $P_1 \bigcap P_2$. When nail 1 comes out of the wall, the loop suddenly finds itself in P_2, and it slips free of nail 2 if it is nulhomotopic in P_2. When nail 2 comes out, the loop slips free of nail 1 if it is nulhomotopic in P_1.

So the problem asks us to find a loop f in $P_1 \bigcap P_2$ such that $i_1 \circ f$ is nulhomotopic and $i_2 \circ f$ is nulhomotopic. In other words, we wish to find $[f] \in \pi_1(P_1 \bigcap P_2, x)$ such that $(i_1)_*[f] \in \pi_1(P_1, x)$ and $(i_2)_*[f] \in \pi_1(P_2, x)$ are both trivial.

Remark: $P_1 \bigcap P_2$ is homotopy-equivalent to the figure-eight space. As we will see shortly in this course, the fundamental group of the figure-eight is the free group on the two generators a and b mentioned above. The effect of $(i_1)_*$ is to send b to 1, and the effect of $(i_2)_*$ is to send a to 1. Thus

$$(i_1)_*[ab\bar{a}\bar{b}] = [a1\bar{a}1] = [a\bar{a}] = 1,$$

and $(i_2)_*[ab\bar{a}\bar{b}] = 1$ symmetrically.

4. Compact-Open Topology

For any topological spaces X and Y, let $C(X,Y)$ denote the set of continuous functions $f : X \to Y$. Endow $C(X,Y)$ with the *compact-open topology* described on pages 285-286 of your book. (When $X = \mathbb{R}$ this is similar to, but not quite the same as, Exam 1 #5.)

For the remainder of this problem, Y is any compact, Hausdorff topological space.

A. For any spaces X and Z, define a map

$$m : C(X,Y) \times C(Y,Z) \to C(X,Z)$$

by $m(f,g) = g \circ f$. Prove that m is continuous.

Answer: We wish to show that the preimage of any open set in $C(X,Z)$ is open in $C(X,Y) \times C(Y,Z)$. Let $C \subseteq X$ be compact and $U \subseteq Z$ be open, so that $S(C,U)$ is a subbasis element for $C(X,Z)$. It suffices to show that $m^{-1}(S(C,U))$ is open. Let $(f,g) \in m^{-1}(S(C,U))$. It suffices to construct a basis element $S(C,W) \times S(D,U)$ in $C(X,Y) \times C(Y,Z)$ such that

$$(f,g) \in S(C,W) \times S(D,U) \subseteq m^{-1}(S(C,U)).$$

(For then any point in $m^{-1}(S(C,U))$ has a neighborhood contained in $m^{-1}(S(C,U))$, and it follows that $m^{-1}(S(C,U))$, being the union of all of these neighborhoods, must be open.)

To begin, $(f,g) \in m^{-1}(S(C,U))$ means that $f : X \to Y$ and $g : Y \to Z$ are continuous and $f(C) \subseteq g^{-1}(U)$. Since g is continuous, $g^{-1}(U)$ is open in Y. Since f is continuous, $f(C)$ is compact in Y (by 26.5). But every compact subset of a Hausdorff space is closed (by 26.3), so $f(C)$ is closed. Thus $f(C)$ and $Y - g^{-1}(U)$ are disjoint closed subsets of Y. Now Y, being compact and Hausdorff, is also normal (by 32.3). Thus there exist disjoint open sets W and V in Y such that $f(C) \subseteq W$ and $Y - g^{-1}(U) \subseteq V$. Let $D = Y - V$. Then D is a closed subset of a compact space, so it is compact (by 26.2). Therefore $S(C,W)\times S(D,U)$ is an open set in $C(X,Y)\times C(Y,Z)$.

Because $f(C)\subseteq W$ we have $f\in S(C,W)$. Also, $D\subseteq g^{-1}(U)$, so $g\in S(D,U)$. Thus $(f,g)\in S(C,W)\times S(D,U)$. Finally, let $(f',g')\in S(C,W)\times S(D,U)$. Then, because $W\subseteq D$,

$$g'(f'(C))\subseteq g'(W)\subseteq g'(D)\subseteq U.$$

Thus $(f',g')\in m^{-1}(S(C,U))$ and $S(C,W)\times S(D,U)\subseteq m^{-1}(S(C,U))$, as desired.

B. For any space Z, define a map $e:Y\times C(Y,Z)\to Z$ by $e(y,g)=g(y)$. Using part A — not some other method — prove that e is continuous. (Free hint: What if X were the one-point space $\{p\}$?)

Answer: Let $X=\{p\}$ be the one-point space. The basic idea is that $C(X,Y)\cong Y$, $C(X,Z)\cong Z$, and the composition map m from Part A reduces to the evaluation map $e:Y\times C(Y,Z)\to Z$.

First, for any point $y\in Y$ define $g_y:X\to Y$ by $g_y(p)=y$. Then the function

$$G:Y\to C(X,Y)$$

defined by $G(y)=g_y$ is a bijection. We show that G is continuous. Let $S(C,U)\subseteq C(X,Y)$ be a subbasis element. Then C is either $\emptyset$ or X. (These are the only two subsets of X, and they are both finite and hence compact.) Then $G^{-1}(S(\emptyset,U))=Y$ (since, for any $y\in Y$, $g_y(\emptyset)=\emptyset\subseteq U$) and $G^{-1}(S(X,U))=U$ (since $g_y(X)\subseteq U\Leftrightarrow y\in U$). Thus $G^{-1}(S(C,U))$ is open and G is continuous. Now we show that G^{-1} is continuous. We have already seen that $G^{-1}(S(X,U))=U$ for any open $U\subseteq Y$. Since G is a bijection, we have $G(U)=S(X,U)$, which is open in $C(X,Y)$. Thus G carries open sets in Y to open sets in $C(X,Y)$, so G^{-1} is continuous. In summary, $G:Y\to C(X,Y)$ is a homeomorphism. Similarly, defining $h_z(p)=z$ gives a homeomorphism $H:Z\to C(X,Z)$.

Now $e:Y\times C(Y,Z)\to Z$ equals the composition

$$Y\times C(Y,Z)\xrightarrow{G\times\mathrm{id}}C(X,Y)\times C(Y,Z)\xrightarrow{m}C(X,Z)\xrightarrow{H^{-1}}Z,$$

because $e(y,f)=f(y)$ and

$$H^{-1}(m((G\times\mathrm{id})(y,f)))=H^{-1}(m(f_y,f))=H^{-1}(f\circ f_y)=f(y).$$

Therefore e, being the composition of continuous functions, is continuous.

C. Let G be the set of homeomorphisms from Y to Y. Then G is a group under composition. Also, G is a subset of $C(Y,Y)$, so it is a topological space, under the subspace topology from the compact-open topology on $C(Y,Y)$. Prove that G is a *topological group*, as defined on page 145.

Answer: We must show three things: the T_1 condition, that the group operation $n:G\times G\to G$ defined by $n(g_1,g_2)=g_2\circ g_1$ is continuous, and that the inversion $i:G\to G$ defined by $i(g)=g^{-1}$ is also continuous.

First, let g_1,g_2 be distinct points in G. So g_1 and g_2 are homeomorphisms of Y, and they differ in value at at least one point: $g_1(y)\neq g_2(y)$ for some $y\in Y$. Since Y is Hausdorff, there exist disjoint neighborhoods U_1, U_2 of $g_1(y)$, $g_2(y)$ in Y. Notice that $\{y\} \subseteq Y$ is compact because it is finite, that g_1 is an element of $S(\{y\}, U_1)$, and that g_2 is not an element of $S(\{y\}, U_1)$. Thus we have found a neighborhood $S(\{y\}, U_1)$ of g_1 that does not contain g_2. Since we can do this for any $g_1 \neq g_2$, this proves that $G - \{g_2\}$ is open and $\{g_2\}$ is closed. Since g_2 was arbitrary, G is T_1.

Second, notice that $n : G \times G \to G$ is the restriction of $m : C(Y, Y) \times C(Y, Y) \to C(Y, Y)$, which is continuous by Part A. So n is continuous.

Third, let $S(C, U)$ be a subbasis element for G. (It is actually a subbasis element for $C(Y, Y)$; I leave it to you to intersect open sets in $C(Y, Y)$ with G.) We wish to show that $i^{-1}(S(C, U))$ is open in G. For starters, since U is open, $Y - U$ is closed and hence compact (by 26.2); also, since C is compact, C is closed (by 26.3) and hence $Y - C$ is open. Therefore $S(Y - U, Y - C)$ is a subbasis element. Now, among homemorphisms $f : Y \to Y$,

$$\begin{array}{rrcl} & f(C) & \subseteq & U \\ \Leftrightarrow & C & \subseteq & f^{-1}(U) \\ \Leftrightarrow & Y - C & \supseteq & Y - f^{-1}(U) \\ & & = & f^{-1}(Y - U). \end{array}$$

Therefore f is an element of $S(C, U)$ if and only if $i^{-1}(f) = i(f) = f^{-1}$ is an element of $S(Y - U, Y - C)$. Thus $i^{-1}(S(C, U)) = S(Y - U, Y - C)$, which is open, and i is continuous.