You may take this exam either as a third midterm exam (from Fri Mar 7 3:20 PM to Mon Mar 10 1:50 PM) or as a final exam (from Wed Mar 12 5:00 PM to Sat Mar 15 5:00 PM). The "or" in that sentence was exclusive-or. Regardless of which period you choose, please do not discuss the exam with anyone, in any way, until Saturday March 15 5:00 PM.

The rules of the exam are identical to the rules of the previous exam. During your chosen time period, you may work on the exam as much as you like. I will try to check my e-mail frequently, and I will be in my office Sunday 2:00-3:00 PM. You may ask clarifying questions for free. If you believe that the statement of a problem is wrong, then you should certainly ask for clarification. You may also ask for hints, which cost you some points, to be decided unilaterally by me as I grade your paper.

The exam is open-book and open-note, which means, precisely:

- You may freely consult all of this class' material: the Munkres textbook, your class notes, your old homework and exams, and the materials on the class web site. If you missed a lecture and need to copy someone else's class notes, do so before either of you begins the exam.
- You may assume all results discussed in class, in the assigned sections of the book, and in the assigned homework problems. You do not have to prove or reprove them on this test. On the other hand, you may not cite results that we have not studied. If you are unsure of whether you are allowed to cite a result, just ask.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may use a computer for these three purposes: viewing the class web site materials, typing up your answers, and e-mailing with me. You may not use a computer or calculator for any other purpose.
- You may not discuss the exam in any way (spoken, written, pantomime, semaphore, etc.) with anyone but me until Saturday March 15 5:00 PM, even if you finish earlier. During the exam you will inevitably see your classmates around campus. Please refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.

Your solutions should be rigorous, self-explanatory, and polished (concise, neat, and well-written, employing complete sentences with punctuation). Always show enough work so that a classmate could follow your solutions. Do not show scratch work, false starts, circuitous reasoning, etc. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Submit your solutions in a single stapled packet, presented in the order they were assigned.

Partial credit is often awarded. Exam grades will be loosely curved — by this I do not mean that there are predetermined numbers of As, Bs, Cs to be awarded, but rather that there are no predetermined scores required for grades A, B, C.

Good luck!

1. Categories

A *category* consists of two things — a collection of *objects* and a collection of *morphisms* — satisfying certain axioms. Historically, the objects have often been sets (possibly with additional structure) and the morphisms have been functions among those sets (that respect the additional structure, if any). For example:

- Let $\mathcal{S}$ be the category of sets and functions. That is, the objects of $\mathcal{S}$ are all sets X, Y, Z, etc., and the morphisms of $\mathcal{S}$ are all functions $f : X \to Y$, $g : Z \to X$, etc.
- Let $\mathcal{G}$ be the category whose objects are groups G, H, etc. and whose morphisms are group homomorphisms $f : G \to H$, etc.
- Let $\mathcal{T}$ be the category whose objects are topological spaces X and whose morphisms are continuous functions $f : X \to Y$.
- Define a *pointed topological space* to be a pair (X, x_0) where x_0 is a chosen point in X. For example, $(\mathbb{R}, 0)$ and $(\mathbb{R}, 1)$ are distinct as pointed topological spaces. Let $\mathcal{P}$ be the category whose objects are pointed topological spaces (X, x_0) and whose morphisms are continuous maps $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$.

Keep these examples in mind as you read the following definition. Also, substitute "set" wherever you see "collection", if you are not worried about the technical definition of the word "set" in mathematics. (You need not be.) Also, substitute "function" for "operator", if you like.

A *category* $\mathcal{C}$ is a collection $\mathrm{obj}(\mathcal{C})$ of *objects* and a collection $\mathrm{mor}(\mathcal{C})$ of *morphisms* with the following properties. First, $\mathcal{C}$ is equipped with

- a *domain* operator $\mathrm{dom} : \mathrm{mor}(\mathcal{C}) \to \mathrm{obj}(\mathcal{C})$,
- a *codomain* operator $\mathrm{cod} : \mathrm{mor}(\mathcal{C}) \to \mathrm{obj}(\mathcal{C})$,
- an *identity* operator $\mathrm{id} : \mathrm{obj}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$; usually id_X is written for $\mathrm{id}(X)$.

The notation "$f : X \to Y$" means that X and Y are objects, f is a morphism, and $\mathrm{dom}(f) = X$ and $\mathrm{cod}(f) = Y$. Let $\mathrm{com}(\mathcal{C})$ be the subcollection of $\mathrm{mor}(\mathcal{C}) \times \mathrm{mor}(\mathcal{C})$ consisting of those pairs (g, f) such that $\mathrm{dom}(g) = \mathrm{cod}(f)$. These are called *composable pairs*. The category $\mathcal{C}$ must possess a *composition* operator

$$\circ : \mathrm{com}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C}).$$

Finally, the following axioms must be satisfied by dom, cod, id, and $\circ$:

(1) For all objects X, $\mathrm{dom}(\mathrm{id}_X) = X = \mathrm{cod}(\mathrm{id}_X)$.

(2) For all pairs (g, f) in $\mathrm{com}(\mathcal{C})$, $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$.

(3) For all morphisms f, g, and h, such that (g, f) and (h, g) are composable,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(4) For all morphisms f,

$$f \circ \mathrm{id}_{\mathrm{dom}(f)} = f = \mathrm{id}_{\mathrm{cod}(f)} \circ f.$$

That's the definition of *category*. To understand it, let's return to the example of the category $\mathcal{S}$. Here the objects are sets X and the morphisms are functions $f : X \to Y$, with $\mathrm{dom}(f) = X$ and $\mathrm{cod}(f) = Y$, as usual. Composition of morphisms just means composition of functions. If X is any set, then id_X has domain and codomain X (by Axiom 1) and is in fact the identity function on X (because it composes with other functions f just like the identity function on X, by Axiom 4).

Our other examples $\mathcal{G}$, $\mathcal{T}$, and $\mathcal{P}$ are similar, but we require our morphisms to be "nice" functions that respect the properties of the objects in question. These are the categories of greatest interest to us in Math 354. Historically, category theory arose out of algebraic topology, and these were among the first categories studied.

In their full generality, categories need not have anything to do with sets and functions. For example, one can regard any group G as a category consisting of a single object and one morphism for each element of G. (How?) All of this may seem rather abstract and artificial, but I assure you that categories arise in (semi)practical fields such as theoretical physics and computer programming language design.

A *covariant functor* from the category $\mathcal{C}$ to the category $\mathcal{D}$ is a mapping F that associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$ and to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism

$$F(f) : F(X) \to F(Y)$$

in $\mathcal{D}$, such that these two axioms hold:

(1) For every object X in $\mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(2) For every composable pair (g, f) in $\mathcal{C}$, $(F(g), F(f))$ is composable in $\mathcal{D}$ and

$$F(g \circ f) = F(g) \circ F(f).$$

A. Prove that π_1 (the fundamental group) is a covariant functor from $\mathcal{P}$ to $\mathcal{G}$. Be very clear about what it does to both objects and morphisms.

B. Define a Cartesian product operation on pointed topological spaces. (Given two pointed topological spaces, your Cartesian product should also be a pointed topological space.)

C. There is also a Cartesian product operation on groups. Show that the π_1 functor respects the Cartesian product. (For this, you may cite *any* theorem in the book, without proof.)

2. Euler Characteristic

On Friday we saw that any compact, connected surface can be triangulated. Let V, E, and F be the number of vertices, edges, and faces (triangles) used in a triangulation. Define the *Euler characteristic* to be

$$\chi = V - E + F.$$

It turns out that any given compact, connected surface X has a well-defined $\chi(X)$, independent of the triangulation chosen; you may assume this fact.

A. Compute χ for the sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$, and real projective plane $\mathbb{RP}^2$.

B. Recall that the *connected sum* $X \# Y$ of two surfaces X and Y is the surface obtained by cutting an open disk (or triangle) out of each and gluing the resulting boundary circles together. Compute $\chi(X \# Y)$ in terms of $\chi(X)$ and $\chi(Y)$.

C. Compute χ for all compact, connected surfaces.

3. Polyhedra

A *polyhedron* is a compact, connected surface in $\mathbb{R}^3$ that is the union of polygons. The polygons are called *faces*, the line segments where they intersect are called *edges*, and the points where the edges intersect are called *vertices*. A polyhedron is said to be *regular* if all of its faces have the same number $n \geq 3$ of sides and all of its vertices are connected to the same number $d \geq 3$ of edges. (Note: Some authors use other, inequivalent definitions of regularity.) For example, a cube is a regular polyhedron. It turns out that any regular polyhedron is homeomorphic to $\mathbb{S}^2$; you may assume this fact.

Let v, e, f be the number of vertices, edges, and faces in any given regular polyhedron.

A. Prove that $v - e + f = \chi(\mathbb{S}^2)$.

B. Prove that $dv = 2e = nf$.

C. What combinations of n and d can exist?

4. Seifert-van Kampen?

The Seifert-van Kampen theorem (70.1, 70.2, etc.) is the largest result of this course (except perhaps for the Urysohn lemma, which we didn't prove). Yet it has not been mentioned in this exam — at least, not explicitly. What problems, if any, rely on the Seifert-van Kampen theorem or on results that we proved using Seifert-van Kampen? Explain thoroughly.

5. Time Spent

Please report the number of hours you spent on this exam, rounded to the nearest hour. Your answer does not affect your grade.