Triangle Inequality: For all real a and b, $|a+b| \le |a| + |b|$.

Inverse Triangle Inequality: For all real a and b, $|a-b| \ge ||a|-|b||$.

Proof. By the triangle inequality,

$$|a| = |a-b+b| \le |a-b| + |b| \;\Rightarrow\; |a-b| \ge |a| - |b|.$$

Symmetrically,

$$|b| = |b-a+a| \le |b-a| + |a| \;\Rightarrow\; |b-a| \ge |b| - |a|.$$

But $|b-a| = |a-b|$, so $|a-b|$ is greater than or equal to both $|a|-|b|$ and $|b|-|a|$.

Convexity of e^x: For all real a and b and all $t \in [0,1]$,

$$e^{ta+(1-t)b} \le te^a + (1-t)e^b.$$

Proof Sketch. Let

$$f(x) = \frac{e^b - e^a}{b-a}(x-a) + e^a$$

be the line through (a, e^a) and (b, e^b). Notice that $ta + (1-t)b$ is in $[a,b]$, and that $e^x \le f(x)$ on $[a,b]$, since $y = e^x$ is convex (meaning concave-up). Thus

$$e^{ta+(1-t)b} \le f(ta + (1-t)b) = te^a + (1-t)e^b.$$

Young's Inequality: Let $p, q > 1$ be real such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real a and b,

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Proof. Let $t = \frac{1}{p}$, so that $1 - t = \frac{1}{q}$. Then, using the convexity of e^x,

$$ab = e^{\log a + \log b} = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \le \frac{1}{p}e^{\log a^p} + \frac{1}{q}e^{\log b^q} = \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Cauchy's Inequality: For all real a and b,

$$ab \le \frac{a^2}{2} + \frac{b^2}{2}.$$

Proof. This is the special case of Young's Inequality with $p = q = 2$. It is also easy to prove directly: $0 \le (a-b)^2 = a^2 - 2ab + b^2 \;\Rightarrow\; 2ab \le a^2 + b^2$.

The following inequalities concern the ℓ^p-norms of vectors $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. In particular, Minkowski's inequality is the triangle inequality for the ℓ^p-norm. All sums $\sum$ are taken over an index i running from 1 to n.

Discrete Hölder's Inequality: Let $p, q > 1$ be real such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \in \mathbb{R}^n$,

$$\sum |a_i b_i| \leq \left(\sum |a_i|^p\right)^{1/p} \left(\sum |b_i|^q\right)^{1/q}.$$

Proof. Let $A = (\sum |a_i|^p)^{1/p}$ and $B = (\sum |b_i|^q)^{1/q}$. Then, using Young's inequality and the triangle inequality,

$$\frac{\sum |a_i b_i|}{AB} = \sum \left| \frac{a_i}{A} \frac{b_i}{B} \right| \leq \sum \left| \frac{1}{p} \left(\frac{a_i}{A}\right)^p + \frac{1}{q} \left(\frac{b_i}{B}\right)^q \right| \leq \sum \left| \frac{1}{p} \left(\frac{a_i}{A}\right)^p \right| + \sum \left| \frac{1}{q} \left(\frac{b_i}{B}\right)^q \right|.$$

This simplifies to

$$\frac{1}{pA^p} \sum |a_i|^p + \frac{1}{qB^q} \sum |b_i|^q = \frac{1}{p} \frac{\sum |a_i|^p}{A^p} + \frac{1}{q} \frac{\sum |b_i|^q}{B^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus $\sum |a_i b_i| = AB$, as desired.

Discrete Minkowski's Inequality: Let $p > 1$ be real. Then for all $a, b \in \mathbb{R}^n$,

$$\left(\sum |a_i + b_i|^p\right)^{1/p} \leq \left(\sum |a_i|^p\right)^{1/p} + \left(\sum |b_i|^p\right)^{1/p}.$$

Proof. By the triangle inequality,

$$\sum |a_i + b_i|^p \leq \sum (|a_i| + |b_i|) \, |a_i + b_i|^{p-1} = \sum |a_i||a_i + b_i|^{p-1} + \sum |b_i||a_i + b_i|^{p-1}.$$

Applying the discrete Hölder's inequality with $q = \frac{p}{p-1}$ (so that $\frac{1}{p} + \frac{1}{q} = 1$) to the first term on the right-hand side rewrites it as

$$\left(\sum |a_i|^p\right)^{1/p} \left(\sum \left(|a_i + b_i|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} = \left(\sum |a_i|^p\right)^{1/p} \left(\sum |a_i + b_i|^p\right)^{\frac{p-1}{p}}.$$

Do the same to the other term and combine the results. Then the inequality is

$$\sum |a_i + b_i|^p \leq \left(\left(\sum |a_i|^p\right)^{1/p} + \left(\sum |b_i|^p\right)^{1/p}\right) \left(\sum |a_i + b_i|^p\right)^{\frac{p-1}{p}}.$$

Divide through by $(\sum |a_i + b_i|^p)^{\frac{p-1}{p}}$ to obtain the result.

These results also hold for $p = 1$ and "$q = \infty$", by *ad hoc* arguments. They also hold if we replace the vectors a, b with (appropriate) functions f, g and the sums $\sum$ with integrals $\int_c^d \dots dx$. They are then statements about L^p-norms on vector spaces of functions, which are studied in courses in functional analysis.