

1. Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 4 & 4 \\ -1 & 1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 14 \\ 0 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 14 \\ 0 \\ 14 \end{bmatrix}.$$

Compute A^{-1} and use it to solve $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{c}$. Clearly mark your answers.

Answer: [I'll not show the work here.]

$$A^{-1} = \begin{bmatrix} -2/7 & 3/14 & -4/7 \\ -2/7 & 3/14 & 3/7 \\ 3/7 & -1/14 & -1/7 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -12 \\ 2 \\ 4 \end{bmatrix}.$$

2. Find the polynomial $y = ax^3 + bx^2 + cx + d$ that passes through the points $(-1, 3)$, $(0, 1)$, $(1, 3)$, and $(2, 4)$. (Note: Most of the points for this problem are earned by setting up the relevant equations. Only solve the equations if you have extra time at the end of the exam.)

Answer: [I'll not show the work here.] $y = \frac{-5}{6}x^3 + 2x^2 + \frac{5}{6}x + 1$.

3. In the Democratic Republic of Pretendland (DRP) there are n major cities, which are called C_1, \dots, C_n (in decreasing order of population). Some of the cities are connected by non-stop high-speed trains; let A be the $n \times n$ adjacency matrix of this train network. Some of the cities are connected by non-stop airplane flights; let B be the $n \times n$ adjacency matrix of this airplane network. Just so we're clear: There is one set of cities here, but there are two separate graphs built on them, with different adjacency matrices.

A. You are planning a trip to the DRP in which you'll be traveling to many of its major cities. How is the matrix $A + B$ useful to you?

Answer: $(A + B)_{ij}$ is 0 if there is no non-stop train or airplane flight from C_i to C_j . It is greater than 0 if there is either a non-stop train or airplane flight. In particular, it is 2 if there is both a non-stop train and a non-stop airplane flight. So $A + B$ helps us determine which cities we can travel between using non-stop trips.

B. Mathematically, what is the difference between $(A + B)^2$ and $A^2 + B^2$? Be sure to show every step of your work.

Answer:

$$\begin{aligned} (A + B)^2 - (A^2 + B^2) &= (A + B)(A + B) - A^2 - B^2 \\ &= A^2 + AB + BA + B^2 - A^2 - B^2 \\ &= AB + BA. \end{aligned}$$

C. Practically, what is the difference between $(A + B)^2$ and $A^2 + B^2$, to your trip?

Answer: $(A^2 + B^2)_{ij}$ gives the number of one-stop (two-leg) train and airplane rides from C_i to C_j . $((A + B)^2)_{ij}$ gives the number of one-stop trips from C_i to C_j , including trips by train then airplane (measured by $(AB)_{ij}$) and trips by airplane then train (measured by $(BA)_{ij}$).

The train-airplane and airplane-train trips measured by $(AB + BA)_{ij}$ are probably not as convenient as the all-train or all-airplane trips measured by $(A^2 + B^2)_{ij}$, since they require movement between a train station and an airport in the intermediary city. A traveller who is primarily concerned with speed and convenience might limit her choices to those measured by $A^2 + B^2$. A traveller who is primarily concerned with cost (or who enjoys meandering or variety) will want to consider train-airplane and airplane-train trips too, hence $(A + B)^2$.

4. After your trip to the DRP, you return to your job as an aerospace engineer. You are working on a spacecraft that, at a crucial point in its maneuvers, rotates in two stages. First the spacecraft rotates about the z -axis of space through an angle α . Three minutes later it rotates about the y -axis of space through an angle of β . Find a matrix that expresses the combined effect of the z -axis rotation followed by the y -axis rotation.

Answer:

$$\begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta & -\sin \alpha \cos \beta & \sin \beta \\ \sin \alpha & \cos \alpha & 0 \\ -\cos \alpha \sin \beta & \sin \alpha \sin \beta & \cos \beta \end{bmatrix}.$$

5. What's a *linear transformation*?

Answer: A linear transformation is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$ and $f(k\vec{x}) = kf(\vec{x})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all $k \in \mathbb{R}$. Equivalently, a linear transformation is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose effect on any vector $\vec{x} \in \mathbb{R}^n$ can be described by multiplying that vector by an $n \times m$ matrix A :

$$f(\vec{x}) = A\vec{x}.$$

[Remark: As we learn more linear algebra, the matrix definition becomes less useful to us; the relationship between linear transformations and matrices becomes more subtle.]

6. Each part A-H is a true/false question, but there are three valid answers: TRUE, FALSE, and PUNT. If you answer PUNT, then you receive half credit. Otherwise, if you answer correctly then you receive full credit, and if you answer incorrectly then you receive no credit. No explanation is necessary. Do not just write T, F, or P; write the entire word, and box your answer. Here, f and g denote linear transformations from \mathbb{R}^n to \mathbb{R}^n , and A , B , and C denote $n \times n$ matrices.

A. Any \vec{x} in $\ker f$ is also in $\ker(g \circ f)$.

Answer: TRUE. If $\vec{x} \in \ker f$, then $f(\vec{x}) = \vec{0}$, so

$$(g \circ f)(\vec{x}) = g(f(\vec{x})) = g(\vec{0}) = \vec{0}.$$

Therefore $\vec{x} \in \ker(g \circ f)$.

B. Any \vec{x} in $\ker(g \circ f)$ is also in $\ker f$.

Answer: FALSE. For example, let f and g be the linear transformations corresponding to the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

respectively. Then $\ker f = \{\vec{0}\}$, while $\ker(g \circ f) = \mathbb{R}^2$.

C. There exists a 2×3 matrix with rank 3.

Answer: FALSE. In the RREF of the matrix, there cannot be more leading 1s than rows, since each row has at most one leading 1.

D. There exists a 3×2 matrix with rank 3.

Answer: FALSE. There cannot be more leading 1s than columns, since each column has at most one leading 1.

E. If $A^2 = 0$, then $A = 0$ (where 0 means the zero matrix).

Answer: FALSE. Consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

F. If $AB = AC$, then $B = C$.

Answer: FALSE. Let A be the same matrix as in the answer to Part E. Let $B = A$. Let C be the zero matrix. Then $AB = AC$ but $B \neq C$.

G. A and $\text{rref}(A)$ have the same kernel.

Answer: TRUE. This is a fundamental fact of linear algebra; indeed, it is the reason why the method of row-reducing matrices to solve linear systems works at all! There are several ways to prove it. The simplest is to prove that each of the elementary row operations preserves the kernel; for then the entire act of reducing to RREF preserves the kernel. There are three elementary row operations, but one of them (swapping two rows) can be accomplished using the other two (how?), so there are really only two elementary row operations to worry about: scaling a row, and adding a multiple of one row to another. Convince yourself that both of these preserve the kernel. [Remark: Parts G and H were assigned as homework: 3.1 #44.]

H. A and $\text{rref}(A)$ have the same image.

Answer: FALSE. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{which reduces to} \quad \text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The image of A is the line through $(1, 1)$ in \mathbb{R}^2 ; the image of $\text{rref}(A)$ is the line through $(1, 0)$.