1. Find a basis for the orthogonal complement of the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.

Answer: Let A be the 3×4 matrix with columns $\vec{u}$, $\vec{v}$, $\vec{w}$. Then the span of $\{\vec{u}, \vec{v}, \vec{w}\}$ is the image of A, and the orthogonal complement of the span is the kernel of $A^\top$ (Fact 5.4.1). After a standard row-reduction process [which I'll omit here], $\ker A^\top$ is spanned by

$$\begin{bmatrix} 1 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}.$$

These are linearly independent, hence a basis for $\ker A^\top$.

2A. Show that the kth power of A is symmetric if k is even and skew-symmetric if k is odd.

Answer: Since A is skew-symmetric, $A^\top = -A$. Thus

$$(A^\top)^k = (-A)^k = (-1)^k A^k.$$

But $(A^\top)^k = (A^k)^\top$ (from Fact 5.3.9a), so we have

$$(A^k)^\top = (-1)^k A^k.$$

If k is even, then $(A^k)^\top = A^k$ so A^k is symmetric; if k is odd, then $(A^k)^\top = -A^k$ so A^k is skew-symmetric.

2B. Show that if n is odd then A cannot be invertible.

Answer: Using Facts 6.2.7, 6.2.4, and 6.1.6,

$$\det A = \det(A^\top) = \det(-A) = \det(-IA) = \det(-I)\det A = (-1)^n \det A.$$

If n is odd, then $\det A = -\det A$, so $\det A = 0$ and A cannot be invertible.

3A. Let $\langle \cdot, \cdot \rangle$ be an inner product defined by a symmetric, positive-definite matrix A as above. Find a condition (like $[T]_{\mathcal{E}}^\top [T]_{\mathcal{E}} = I$ — but it will be different from this) on the matrix $[T]_{\mathcal{E}}$ that determines whether or not a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$.

Answer: First,

$$\begin{aligned} \langle T(\vec{v}), T(\vec{w}) \rangle &= ([T]_{\mathcal{E}}\vec{v})^\top A \, ([T]_{\mathcal{E}}\vec{w}) \\ &= \vec{v}^\top [T]_{\mathcal{E}}^\top A \, [T]_{\mathcal{E}}\vec{w}, \end{aligned}$$

while $\langle \vec{v}, \vec{w} \rangle = \vec{v}^\top A \, \vec{w}$. Therefore orthogonality means

$$\vec{v}^\top [T]_{\mathcal{E}}^\top A \, [T]_{\mathcal{E}}\vec{w} = \vec{v}^\top A \, \vec{w}$$

for all vectors $\vec{v}, \vec{w}$. Because this equation holds for all $\vec{v}$ and $\vec{w}$, it must be that

$$[T]_{\mathcal{E}}^\top A \, [T]_{\mathcal{E}} = A.$$

3B. Find a change-of-basis formula that relates $[\langle \cdot, \cdot \rangle]_{\mathcal{B}}$ to $[\langle \cdot, \cdot \rangle]_{\mathcal{E}}$.

Answer: Recall the change-of-basis formula for vectors:

$$[\vec{v}]_{\mathcal{B}} = [\mathcal{E}]_{\mathcal{B}} [\vec{v}]_{\mathcal{E}}.$$

Working from that formula, we have

$$\begin{aligned} [\vec{v}]_{\mathcal{E}}^{\top} \, [\langle \cdot, \cdot \rangle]_{\mathcal{E}} \, [\vec{w}]_{\mathcal{E}} &= \langle \vec{v}, \vec{w} \rangle \\ &= [\vec{v}]_{\mathcal{B}}^{\top} \, [\langle \cdot, \cdot \rangle]_{\mathcal{B}} \, [\vec{w}]_{\mathcal{B}} \\ &= \left([\mathcal{E}]_{\mathcal{B}} [\vec{v}]_{\mathcal{E}}\right)^{\top} \, [\langle \cdot, \cdot \rangle]_{\mathcal{B}} \, \left([\mathcal{E}]_{\mathcal{B}} [\vec{w}]_{\mathcal{E}}\right) \\ &= [\vec{v}]_{\mathcal{E}}^{\top} [\mathcal{E}]_{\mathcal{B}}^{\top} \, [\langle \cdot, \cdot \rangle]_{\mathcal{B}} \, [\mathcal{E}]_{\mathcal{B}} [\vec{w}]_{\mathcal{E}}. \end{aligned}$$

Since this equation holds for all vectors $\vec{v}$ and $\vec{w}$ it follows that

$$[\langle \cdot, \cdot \rangle]_{\mathcal{E}} = [\mathcal{E}]_{\mathcal{B}}^{\top} \, [\langle \cdot, \cdot \rangle]_{\mathcal{B}} \, [\mathcal{E}]_{\mathcal{B}}.$$

4A. Which parts of the definition of inner product does $\langle \cdot, \cdot \rangle$ satisfy, and which parts does it not satisfy?

Answer: It satisfies the addition, scalar multiplication, and symmetry properties. It does not satisfy the positive-definiteness property. [I'll omit the details here.]

4B. For which vectors $\vec{v} \in \mathbb{R}^2$ is $||\vec{v}||$ defined? For which $\vec{v}$ is it 0? For which $\vec{v}$ is it 1? Answer these questions both in words/equations and in a detailed sketch of $\mathbb{R}^2$.

Answer: Since

$$||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{v_1 v_1 - v_2 v_2} = \sqrt{v_1^2 - v_2^2},$$

$||\vec{v}||$ is defined if and only if $v_1^2 \geq v_2^2$. It is 0 exactly when $v_1^2 = v_2^2$, which occurs along the lines $v_1 = \pm v_2$. It is 1 exactly when $v_1^2 - v_2^2 = 1$, which occurs along a hyperbola that has the two aforementioned lines as asymptotes. [I'll omit the picture here.]

4C. Describe the $\langle \cdot, \cdot \rangle$-rotation matrices in terms of $\cosh$ and $\sinh$.

Answer: By the same argument as we used in 3A, a matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

preserves $\langle \cdot, \cdot \rangle$ if and only if

$$B^{\top} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now

$$B^{\top} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B = \begin{bmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{bmatrix},$$

so B is special orthogonal with respect to $\langle\cdot,\cdot\rangle$ if and only if these four equations are satisfied:

$$\begin{aligned} a^2 - c^2 &= 1, & (1)\\ ab - cd &= 0, & (2)\\ b^2 - d^2 &= -1, & (3)\\ ad - bc &= 1. & (4) \end{aligned}$$

Given numbers a, b, c, d satisfying the equations, let $x = \operatorname{arcsinh} c$. Then equation (1) becomes $a^2 - \sinh^2 x = 1$. One solution is $a = \cosh x$, and the other must be $a = -\cosh x$. Similarly, let $y = \operatorname{arcsinh} b$; then $d = \pm\cosh y$ by equation (3). We have already deduced that B must be of the form

$$B = \begin{bmatrix} (-1)^k \cosh x & \sinh y \\ \sinh x & (-1)^\ell \cosh y \end{bmatrix}.$$

Now the determinant of B is

$$\det B = (-1)^{k+\ell} \cosh x \cosh y - \sinh x \sinh y.$$

If $k+\ell$ is odd, then (by equation (4))

$$1 = \det B = -\cosh x \cosh y - \sinh x \sinh y = -\cosh(x+y).$$

This is impossible, since $\cosh z$ is positive for all numbers z. Therefore $k+\ell$ is even, so $(-1)^k = (-1)^\ell$ and

$$1 = \det B = \cosh x \cosh y - \sinh x \sinh y = \cosh(x-y).$$

But $\cosh z = 1$ if and only if $z = 0$. Thus $x = y$ and B must be of the form

$$B = \begin{bmatrix} \pm\cosh x & \sinh x \\ \sinh x & \pm\cosh x \end{bmatrix}$$

(with the two "$\pm$" matching). For the "$-$" variant, notice that

$$\begin{bmatrix} -\cosh x & \sinh x \\ \sinh x & -\cosh x \end{bmatrix} = -\begin{bmatrix} \cosh x & -\sinh x \\ -\sinh x & \cosh x \end{bmatrix} = -\begin{bmatrix} \cosh(-x) & \sinh(-x) \\ \sinh(-x) & \cosh(-x) \end{bmatrix}.$$

So in the end B must be of the form

$$B = \pm\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

for some number θ. It is now easily checked that all matrices of this form satisfy equations (1) through (4). Therefore these matrices, and only these matrices, are special orthogonal with respect to $\langle\cdot,\cdot\rangle$.

[Remark: This is certainly the most difficult argument on the exam. Most students were able to demonstrate that matrices of the form

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

are special orthogonal with respect to $\langle \cdot, \cdot \rangle$, and this earned 5/9 points. If they also caught the "$\pm$" variation, then they earned 6/9 points. To earn all 9/9 points, they were required to show that there are no other special orthogonal matrices, as above.]

5. Using techniques from this course, find the route for the water main that minimizes the total length of small pipe that the landowners must use, in a least-squares sense.

Answer: We want to fit a line $y = c + dx$ to the given (x, y) data. Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 5 \\ 7 \\ 6 \\ 2 \\ 8 \\ 6 \end{bmatrix}.$$

Then the data are on $y = c + dx$ if and only if $A\vec{v} = \vec{b}$. There are no numbers c and d to make this happen — after all, there is no line through the given data — but the best approximate solution (in the least-squares sense) is given by

$$\begin{bmatrix} c \\ d \end{bmatrix} = \left(A^\top A\right)^{-1} A^\top \vec{b} = \begin{bmatrix} \frac{177}{35} \\ \frac{4}{35} \end{bmatrix}.$$

Therefore the best-fit line is

$$y = \frac{177}{35} + \frac{4}{35}x \approx 5.05714 + 0.114286x.$$

6A. Start with a spacecraft upright and pointing along the y-axis. Draw what the spacecraft looks like after it yaws $\pi/2$ and then pitches $\pi/2$. In a separate picture, draw what the spacecraft looks like after it pitches $\pi/2$ and then yaws $\pi/2$.

Answer: [I'll omit the pictures here.] The yawed-then-pitched spacecraft is pointing along the z-axis, with its tail fin pointing along the x-axis. The pitched-then-yawed spacecraft is pointing along the $-x$-axis, with its tail fin pointing along the $-y$-axis.

6B. Again start with a spacecraft upright and pointing along the y-axis. The spacecraft is going to yaw θ and then pitch ϕ. Find a matrix that expresses the yaw. Find a matrix that expresses the pitch from the yawed position. Find a matrix that expresses the net effect of the yaw followed by the pitch.

Answer: The initial yaw is a rotation about the z-axis, hence given by

$$Y = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now pitch from the unyawed position would be rotation about the x-axis, namely

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

but this is not the same as pitch from the yawed position. The yaw Y takes the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ to

$$\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Using this basis, we can construct a change of coordinates that expresses the pitch P from the yawed position as

$$P = YQY^{-1}.$$

Finally, the net effect of the yaw followed by the pitch from the yawed position is

$$PY = YQY^{-1}Y = YQ.$$

[Remark: This may be counterintuitive, in that it ends up looking like a pitch transformation Q followed by a yaw transformation Y, not the other way around. In pitch/yaw/roll the rotation axes are attached to the object, instead of fixed to the space around the object. The rotation axes get rotated whenever the object rotates. As we see here, the practical difference is that everything happens in an inverse/backward manner. If you plug in $\theta = \pi/2$ and $\phi = \pi/2$ and compute what YQ does to the standard basis vectors, you will see that the formulas here do indeed agree with Part A.]