You have 150 minutes. You may not use any calculator or "cheat sheet".

Show your work. Useful work often earns partial credit; correct answers with no explanation often earn no credit.

If you have forgotten a key formula or concept, or just don't know how to begin a problem, then you may ask for a hint. The value of the hint is decided by me as I grade your paper.

Good luck!

1. An electrical circuit is a network of wires with current (electrical charge) flowing along each wire. Kirchhoff's current law says that at each junction in the circuit, the total current flowing into the junction equals the total current flowing out of the junction. Pictured below is a circuit of seven wires and three junctions. The currents (in milliamperes) are known on four of the wires and unknown on three of the wires. Find all possible values of the unknown currents.
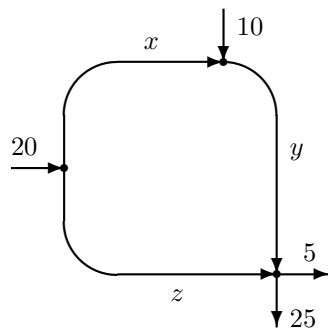
2. Compute the volume of the parallelepiped in $\mathbb{R}^3$ defined by the vectors

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

3. Let A be any $n \times n$ skew-symmetric matrix. Show that the kth power of A is symmetric if k is even and skew-symmetric if k is odd.

4. Diagonalize the matrix A below. That is, find a diagonal D and an invertible S such that $A = SDS^{-1}$.

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -2 & -1 & -2 \\ 1 & -1/2 & -1 \end{bmatrix}.$$

5. No explanation is needed on this problem. Find a single matrix A that satisfies all of these properties:

- 4 appears three times as a root of the characteristic polynomial, and the eigenspace associated to 4 is three-dimensional.

- 3 appears three times as a root of the characteristic polynomial, and the eigenspace associated to 3 is two-dimensional.

- 2 appears three times as a root of the characteristic polynomial, and the eigenspace associated to 2 is one-dimensional.

Also, for your chosen matrix A, find an eigenvector associated to the eigenvalue 2.

6. Find an orthonormal basis for the image of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

7. In 3D computer graphics, an object is described as a list of polygons. Each polygon is described by its vertices, which are points in $\mathbb{R}^3$. For a variety of reasons, it is also useful to give each polygon a *normal vector* $\vec{n}$ — meaning a unit vector, perpendicular to the polygon, that points "out" from the object. See the cube model below left.

Frequently (as in, millions of times per second) we wish to transform our polygons by a 3×3 matrix A. Transforming the vertices of a polygon is simple; we just apply A to each vertex. Transforming the normal vector is more subtle; if we just apply A to the normal vector $\vec{n}$, then the resulting vector $A\vec{n}$ may no longer be perpendicular to the polygon. See the transformed cube below right.
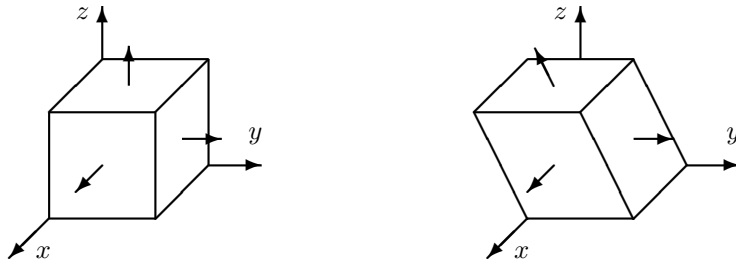
A. Show that if $\vec{n}$ is perpendicular to a given polygon, then $\left(A^{-1}\right)^\top \vec{n}$ is perpendicular to the transformed polygon.

B. For some matrices A, transforming normal vectors by A is okay; it produces the same result as transforming them by $\left(A^{-1}\right)^\top$. Which matrices am I thinking of?

8. Suppose that A is $n \times m$, $\vec{b}$ is $n \times 1$, and $A\vec{x} = \vec{b}$ has no solution. What does it mean, to find the *least squares solution* to $A\vec{x} = \vec{b}$? Explain in words and formulas.

9. Give an example of a vector space, and briefly explain why it is a vector space. The more exotic your example, the better; $\mathbb{R}^n$ earns no points.

10. Each part A-N of this problem is a true/false question, but there are three valid answers: TRUE, FALSE, and PUNT. If you answer PUNT, then you receive half credit. Otherwise, if you answer correctly then you receive full credit, and if you answer incorrectly then you receive no credit. No explanation is necessary. Do not just write T, F, or P; write the entire word, and box your answer.

　　　All matrices are assumed to have real entries.

A. Every $n \times n$ matrix has n real eigenvalues, if we count them according to their algebraic multiplicity.

B. Every $n \times n$ matrix has n real eigenvalues, counted according to their geometric multiplicity.

C. Every $n \times n$ matrix has n complex eigenvalues, counted according to their algebraic multiplicity.

D. Every $n \times n$ matrix has n complex eigenvalues, counted according to their geometric multiplicity.

E. Every symmetric $n \times n$ matrix has n eigenvalues, counted according to their geometric multiplicity.

F. If A is similar to B, then A and B must have the same determinant.

G. If A is similar to B, then A and B must have the same trace.

H. If A is similar to B, then A and B must have the same eigenvalues.

I. If A is similar to B, then A and B must have the same eigenvectors.

J. If $T : V \to W$ is a linear transformation and $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is linearly independent, then $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ must be linearly independent.

K. If $T : V \to W$ is a linear transformation and $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is linearly dependent, then $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ must be linearly dependent.

L. If an $n \times n$ matrix has nonzero determinant, then it must be invertible.

M. If an $n \times n$ matrix is not invertible, then it must have at least one eigenvector.

N. If an $n \times n$ matrix is diagonalizable, then it must have n distinct eigenvalues.

11. Let A be any $n \times n$ matrix. Let

$$f_A(\lambda) = \det(A - \lambda I) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + c_{n-2} \lambda^{n-2} + \cdots + c_1 \lambda + c_0$$

be the characteristic polynomial of A. Let F be the matrix that results when we plug A into its own characteristic polynomial; to be precise,

$$F = c_n A^n + c_{n-1} A^{n-1} + c_{n-2} A^{n-2} + \cdots + c_1 A + c_0 I.$$

A. Let $\vec{v}$ be an eigenvector of A. Show that $F\vec{v} = \vec{0}$.

B. Assume that A is diagonalizable. Show that F is the zero matrix.