1. Every morning thousands of people drive from town T and village V to the city C to work, along the eastbound roads depicted in the map. Each road is labelled with its load (the number of cars on the road, on a typical morning, in thousands). Write a system of linear equations for the unknown loads. You do not need to solve the system.

Answer: At each node (intersection), the number of cars entering the node must equal the number of cars exiting the node. Therefore

$$\begin{aligned} 10 &= x_1 + x_2, \\ 5 + x_2 &= x_3 + x_5, \\ x_1 + x_3 &= x_4, \\ x_4 + x_5 &= 15. \end{aligned}$$

2. Find all solutions of $\begin{bmatrix} 0 & 3 & 1 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 2 & 1 & 1 \\ -4 & 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -4 \\ 28 \end{bmatrix}$.

Answer: [I'll omit some of the work; you should not.] The augmented matrix

$$\begin{bmatrix} 0 & 3 & 1 & 0 & 5 \\ 1 & 2 & -1 & 1 & 3 \\ 2 & 2 & 1 & 1 & -4 \\ -4 & 4 & -3 & -1 & 28 \end{bmatrix}$$

reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 6/11 & -37/11 \\ 0 & 1 & 0 & 1/11 & 25/11 \\ 0 & 0 & 1 & -3/11 & -20/11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore x_4 is the only free variable, and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -6/11 \\ -1/11 \\ 3/11 \\ 1 \end{bmatrix} + \begin{bmatrix} -37/11 \\ 25/11 \\ -20/11 \\ 0 \end{bmatrix}.$$

3 A. Let t be any real number, and let $A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$. Compute A^{-1}.

Answer: [You can do this either by the

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

formula that we proved in class, or by your own Gaussian elimination (which is how we proved that formula).] The answer is

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

B. Explain A and A^{-1} geometrically, in words and pictures. Why does your answer to Part A make sense?

Answer: [I'll omit the pictures. You should not.] The original matrix A represents a counterclockwise rotation by t radians. The inverse A^{-1} represents the inverse transformation — the rotation that undoes the A-rotation — so it must be clockwise rotation by t radians, or, equivalently, counterclockwise rotation by $-t$ radians. Replacing t with $-t$ in the original rotation matrix we get

$$\begin{bmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

which matches A^{-1} as computed above. So the A^{-1} above really is counterclockwise rotation by $-t$ radians, as we expected.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Let L be any line in $\mathbb{R}^2$. The function f sends the line L to some set of points in $\mathbb{R}^2$; call this image set $f(L)$.

A. Assume that $\ker f = \left\{\vec{0}\right\}$. Prove that $f(L)$ is also a line.

Answer: [This was an assigned homework problem, 2.1 #37. For variety I give a different solution here.] Let $\vec{x}$ be any point on L, and let $\vec{v}$ be any nonzero vector pointing along L. Then all points on L are of the form $\vec{x} + t\vec{v}$, and only points on L are of this form. Now apply f:

$$f(\vec{x} + t\vec{v}) = f(\vec{x}) + f(t\vec{v}) = f(\vec{x}) + tf(\vec{v}),$$

because f is a linear transformation. This describes the line through the point $f(\vec{x})$ in the direction of $f(\vec{v})$. (And $f(\vec{v}) \neq \vec{0}$ because $\vec{v}$, being nonzero, is not in the kernel of f.)

B. What happens if you don't assume that $\ker f = \left\{\vec{0}\right\}$?

Answer: If $\ker f \neq \{\vec{0}\}$, then it is a line or the whole plane, and it may contain the line L in question. Then $f(L)$ is not a line but just the single-point set $\{\vec{0}\}$. In other words, the proof above breaks down because we cannot be certain that $f(\vec{v})$ is nonzero.

5. Let $\vec{u}$ be a unit vector in $\mathbb{R}^3$. Let L be the line containing $\vec{u}$ (and the origin). Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the (orthogonal) projection of $\mathbb{R}^3$ onto this line.

A. What properties must f satisfy? (That is, how might you check your answer to Part B?)

Answer: [Depending on the properties you listed, you did not need to list all of these to receive full credit. Some students listed the properties of a linear transformation, but those cannot be used to check the answer to Part B, because any matrix you write will satisfy those, because all matrices represent linear transformations.] The image of f is L. In fact, for any vector $\vec{v}$ in L, $f(\vec{v}) = \vec{v}$. The kernel of f is the plane through the origin that is perpendicular to L. That is, $f(\vec{v}) = \vec{0}$ if and only if $\vec{v}$ is perpendicular to $\vec{u}$. Thus f cannot be invertible. Also, $f \circ f = f$, since projecting twice does nothing more than projecting once; in matrix notation, this is $A_f^2 = A_f$. For any input vector $\vec{v}$, f outputs a vector parallel to $\vec{u}$, with magnitude equal to the cosine of the angle θ between $\vec{v}$ and $\vec{u}$, times the magnitude of $\vec{v}$. That is, $f(\vec{v}) = (\vec{v} \cdot \vec{u})\vec{u} = |\vec{v}| \cos\theta$.

B. Find the matrix for f, in terms of u_1, u_2, u_3. (If you cannot do this for arbitrary $\vec{u}$, then do some examples for partial credit.)

Answer: [This was an assigned homework problem, 2.2 #14. For the explanation, see pages 57-59.] The answer is

$$A_f = \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2u_2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3u_3 \end{bmatrix}.$$

6. Each part A-H is a true/false question, but there are three valid answers: TRUE, FALSE, and PUNT. If you answer PUNT, then you receive half credit. Otherwise, if you answer correctly then you receive full credit, and if you answer incorrectly then you receive no credit. No explanation is necessary. Do not just write T, F, or P; write the entire word, clearly.

A. The rank of a matrix cannot exceed its number of rows.

Answer: True. [The rank is the number of leading 1s in the RREF of the matrix. Each row can have at most one leading 1.]

B. For any two vectors $\vec{x}$, $\vec{y}$ in $\mathbb{R}^2$, any vector in $\mathbb{R}^2$ is a linear combination of $\vec{x}$ and $\vec{y}$.

Answer: False. [If $\vec{x}$ and $\vec{y}$ are parallel, then not every vector in $\mathbb{R}^2$ will be a linear combination of them. This is precisely the idea of linear independence.]

C. (Let A be $p \times q$.) If the rank of A equals the number of columns in A, then for every $\vec{b}$ in $\mathbb{R}^p$ there is a unique solution to $A\vec{x} = \vec{b}$.

Answer: False. [Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.]$$

D. (Let A, B be $n \times n$.) If A and B are diagonal, then they must commute.

Answer: True. [Try it for 2×2 and 3×3 matrices, and you'll get the idea.]

E. (Let A, B be $n \times n$.) If A and B are invertible, then $A + B$ must be invertible as well.

Answer: False. [Let $A = I$ and $B = -I$.]

F. (Let A, B be $n \times n$.) If A and B are invertible, then AB must be invertible as well.

Answer: True. [In fact, we have seen that $(AB)^{-1} = B^{-1}A^{-1}$.]

G. (Let A be $n \times n$.) A and A^2 must have the same image.

Answer: False. [Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.]

H. (Let A be $n \times n$.) A and A^2 must have the same kernel.

Answer: False. [Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.]