# Carleton College Math 232, Winter 2009, Final Exam

You have 150 minutes.

You may not use any notes or calculator.

Except on TRUE/FALSE/PUNT questions, always show your work and explain your answers. Good work often earns partial credit. A correct answer with no explanation often earns little or no credit.

If you have no idea how to solve a problem, or if you have forgotten a key formula that you think you need to know, then you may ask me for a hint. The hint will cost you some points (to be decided by me as I grade your paper), but might help you earn more points overall.

Good luck.

**1**. What is $A^\top$, if $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$?

**2A**. Let $a$ and $b$ be real numbers. Compute all eigenvalues and eigenvectors of $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

**2B**. We have seen several important examples of matrices of that form. Describe them.

**3**. Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be a linear transformation. Prove that $\ker T$ and $\operatorname{im} T$ are subspaces (of what?).

Let $V$ be any finite-dimensional vector space. Let $V^*$ be the set of all linear transformations $f : V \to \mathbb{R}$; this $V^*$ is called the *dual space* of $V$. (This concept appeared on Exam 2 with different notation and wording.) Assume that $V$ comes with an inner product $\langle \cdot, \cdot \rangle$. For any element $w$ of $V$, define a function $f_w : V \to \mathbb{R}$ by $f_w(v) = \langle v, w \rangle$.

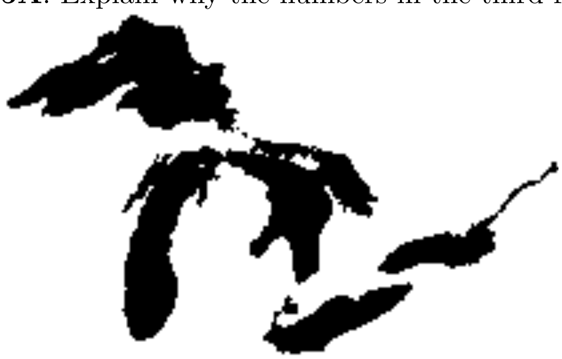
**4A**. Prove, for any element $w$ of $V$, that $f_w$ is an element of $V^*$.

**4B**. Prove, for any element $f$ of $V^*$, that there is some element $w$ of $V$ such that $f = f_w$.

Let $S, M, H, E, O$ denote the concentrations of a certain pollutant in lakes Superior, Michigan, Huron, Erie, and Ontario, respectively. All of the lakes already contain some of the pollutant. Each year, more pollutant is added to Lake Superior by industrial activity along its shores; also, the pollutant mixes among the lakes as they slowly drain eastward into the Atlantic Ocean. While researching the topic you come across the following discrete dynamical system model for the dispersion of the pollutant:

$$\begin{bmatrix} S_{k+1} \\ M_{k+1} \\ H_{k+1} \\ E_{k+1} \\ O_{k+1} \end{bmatrix} = \begin{bmatrix} 1.1 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.8 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.7 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} S_k \\ M_k \\ H_k \\ E_k \\ O_k \end{bmatrix}.$$

**5A**. Explain why the numbers in the third row of the $5 \times 5$ matrix are reasonable.

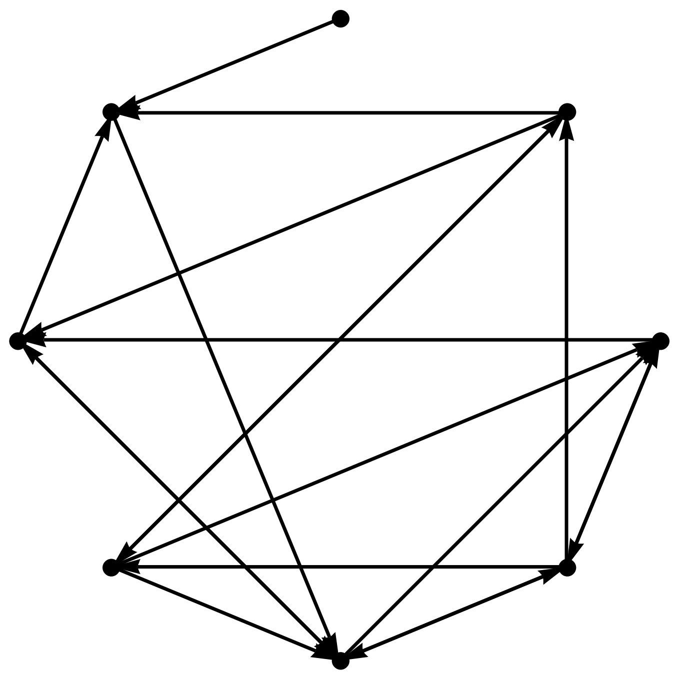


**5B**. To save you time, here are five eigenvectors for the $5 \times 5$ matrix (approximately). What is the long-term trend for the distribution of the pollutant among the Great Lakes?

$$\vec{v}_1 = \begin{bmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.48 \\ -0.89 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 1.00 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0.97 \\ 0.00 \\ 0.24 \\ 0.08 \\ 0.03 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0.00 \\ 0.00 \\ 0.41 \\ 0.41 \\ 0.82 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 0.00 \\ 0.59 \\ 0.59 \\ 0.39 \\ 0.39 \end{bmatrix}.$$

The World Wide Web consists of a billion or more *web pages*, with one-way connections, called *links*, among those pages. In other words, the Web is a giant directed graph. (See figure for example.) When you use Google to search the Web, Google finds all pages relevant to your search but shows you only the most valuable ones. Roughly, a page is valuable if many other pages link to it — especially if those pages are themselves valuable. More precisely, the *value of a link* from page $i$ to page $j$ is the value of page $i$ divided by the number $n_i$ of links on page $i$, and the *value of a page* is the sum of the values of all of the links to that page.

**6**. In the small example below, let $x_i$ be the value of page $i$. Write (but don't solve) a system of equations in standard form, whose solution tells us the values of all of the pages.



**7**. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. What criteria could you use to test whether $f$ is invertible? List all criteria you can think of.

**8**. Give a matrix that has eigenvalues 5, $-1$, and 2 with algebraic multiplicities 2, 3, and 1 (respectively) and geometric multiplicities 2, 2, and 1 (respectively). You do not need to justify your answer.

Each part of this problem is a true/false question, but there are three valid answers: TRUE, FALSE, and PUNT. If you answer PUNT, then you receive half credit. Otherwise, if you answer correctly then you receive full credit, and if you answer incorrectly then you receive no credit. No explanation is necessary. Do not just write T, F, or P; write the entire word, clearly.

**9A**. If an $n \times n$ matrix is diagonalizable, then it must have $n$ distinct eigenvalues.

**9B**. For a rotation of $\mathbb{R}^3$, all real eigenvalues must be positive.

**9C**. If $A$ is $n \times n$, then $A$ and $e^A$ must have the same eigenvectors.

**9D**. If $A$ is a symmetric matrix, $\vec{v}$ and $\vec{w}$ are eigenvectors of $A$, and $\vec{v}$ and $\vec{w}$ are not parallel, then $\vec{v}$ and $\vec{w}$ must be perpendicular.

**9E**. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\mathcal{B}$ and $\mathcal{C}$ are two bases of $\mathbb{R}^n$, then the matrices $[f]_{\mathcal{B}}$ and $[f]_{\mathcal{C}}$ must have the same eigenvalues.

**9F**. For any $n \times m$ matrix $A$, the projection of $\mathbb{R}^m$ onto the image of $A$ is given (with respect to the standard basis on $\mathbb{R}^m$) by $A(A^\top A)^{-1}A^\top$.

**9G**. Any linear system of equations $A\vec{x} = \vec{b}$ has at least one solution.

**9H**. If a subspace $V$ of $\mathbb{R}^3$ contains none of the standard basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, then $V$ must be $\{\vec{0}\}$.

**9I**. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (3x-2y+1, 2x+6y)$ is a linear transformation.

**9J**. If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space $V$, then $\langle v_1+v_2, w_1+w_2 \rangle = \langle v_1, w_1 \rangle + \langle v_1, w_2 \rangle + \langle v_2, w_1 \rangle + \langle v_2, w_2 \rangle$.

**9K**. If $A$ and $B$ are $n \times n$, then $(A + B)^2 = A^2 + 2AB + B^2$.

**9L**. If $A$ and $B$ are both $n \times m$, then $\text{rank}(A + B) = \text{rank } A + \text{rank } B$.

**9M**. If $A$ and $B$ are similar, then $\text{rank } A = \text{rank } B$.

**9N**. Let $V$ be a finite-dimensional vector space and $f : V \to V$ a linear transformation. Then all matrix representations of $f$ have the same trace.

**9O**. An $n \times n$ matrix $A$ is orthogonal if and only if its columns form an orthogonal basis of $\mathbb{R}^n$.

**9P**. If an $n \times n$ matrix $A$ has $n$ eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (counted according to algebraic multiplicity), then $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$.