It is difficult to learn an abstract idea such as *vector space* abstractly; it is much easier to learn by examples that reinforce the abstract idea. Whenever you encounter a new mathematical idea, you should work out some examples to familiarize yourself with it. We do this some in class, and I expect you to do more of it on your own. To help you get started, I have prepared this worksheet. (Revised 2009 February 10.)

We begin with some classic linear transformations. The basics are covered in Section 2.2.

1. Projections: Find the matrix (with respect to the standard basis) for the projection of $\mathbb{R}^2$ onto the x-axis. The y-axis? Any other line? Now find the matrix for the projection of $\mathbb{R}^3$ onto the x-axis. The y-axis? The z-axis? Any other line? The x-y-plane? The y-z- and z-x-planes? Any other plane? Why can I ask more projection questions about $\mathbb{R}^3$ than about $\mathbb{R}^2$? What if we go up to $\mathbb{R}^4$? What kinds of projections are possible? Describe them all. In general, what can you say about the projections of $\mathbb{R}^n$ for larger n? Work out examples.

2. Reflections: Find the matrix (with respect to the standard basis) for reflection of $\mathbb{R}^2$ about the x-axis. The y-axis? Any other line? Now do the same for $\mathbb{R}^3$, $\mathbb{R}^4$, and $\mathbb{R}^n$, as you did for projections. Every question that I asked about projections can also be asked about reflections. More explicitly, there is a one-to-one correspondence between projections and reflections. Explain this statement.

3. Rotations: Recall the matrix (with respect to the standard basis) for counterclockwise rotation of $\mathbb{R}^2$ by θ radians. Try some standard angles such as $\theta = 0, \pi/4, \pi/2, \pi$, etc. and check that the matrices makes sense. How do you describe clockwise rotation by θ radians? Check that the matrices for counterclockwise and clockwise rotation by θ radians are inverses. (This was on Exam 1.) Now let's go up to $\mathbb{R}^3$. Here rotations are famously more complicated, so we need to state some conventions. When we say "rotation about an axis $\vec{v}$ through an angle θ", we mean counterclockwise rotation according to the right-hand rule. (Let me know if you want this explained.) Now find the matrix for counterclockwise rotation about the z-axis through θ radians. Check your answer for $\theta = 0, \pi/4, \pi/2, \pi$, etc. Now do the same for the y-axis and x-axis. (If you want serious practice, then try to find the matrix for rotation about any unit $\vec{u}$ through any angle θ. This will be easier once we've studied Sections 5.1-5.3.)

4. Diagonal Matrices: Diagonal matrices are nice. Work out examples of 2×2 and 3×3 diagonal matrices, visualizing how they deform a box aligned with the coordinate axes. For example, how does the volume of the box change when it is deformed by the matrix? Under what circumstances are projections, reflections, and rotations diagonal? A *scalar matrix* is a diagonal matrix whose diagonal entries are all equal. Work examples to see how scalar matrices deform space. Prove that any $n \times n$ scalar matrix commutes with any other $n \times n$ matrix.

5. Nondiagonal Matrices: Most matrices are not diagonal, of course, but later in the course (Chapter 7) we will learn that they are all "almost diagonal". The poster child for this behavior

is the shear matrix

$$\begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix}.$$

Try a few values for g, to make sure you understand its behavior.

6. Translations (Are Not): Historically, translations and rotations have been significant in mathematics because they are the congruences in Euclidean geometry. (If you rotate or translate a shape, then the result is considered congruent to the original; you haven't "changed anything".) However, translations are not linear transformations. Make sure you understand why.

Vector spaces and their *subspaces* — particularly the *kernel* and *image* of a linear transformation — are crucial; you need to be good at them. The concepts of *linear combination*, *spanning*, *linear independence*, *basis*, *dimension* are also fundamental.

7. List all examples of vector spaces that we have seen so far in the course. Understand why they are vector spaces; in particular, be very clear about what 0 is, what addition is, and what scalar multiplication is in each one.

8. For each vector space, describe all (or at least some) of its subspaces. Also find a subset that is not a subspace.

9. Find a 2×2 matrix such that the kernel (of the the linear tranformation $\mathbb{R}^2 \to \mathbb{R}^2$ that it represents with respect to the standard basis) is 1-dimensional. Is 2-dimensional? Now answer the same questions for 3×3 matrices. Now answer all of the same questions with "image" replacing "kernel". This is warm-up for the next problem.

10. For each subspace of each vector space, find some linear transformation that has that subspace as its kernel. Also find some linear transformation that has that subspace as its image.

As the course goes on I'll add more examples for you to work out.