A. Assume for the sake of contradiction that A is regular. Then by the pumping lemma there exists a pumping length p, such that for the string $b^p a^{p+1} \in A$, there exist strings x, y, z satisfying $b^p a^{p+1} = xyz$, $|xy| \leq p$, $|y| \geq 1$, and $xy^i z \in A$ for all $i \geq 0$. But $y = b^k$ for some k satisfying $1 \leq k \leq p$. So $xyyz = b^{p+k} a^{p+1}$ is not a string in A. This contradiction shows that A is not regular.

For a CFG, I think that $S \to a|aS|aSb|bSa|abS|baS$ works. (Checking these is not easy. In grading, I gave full credit to a CFG, if I couldn't quickly detect a defect in it.) For a PDA, simply take any CFG and apply our algorithm for converting a CFG to a PDA.

B. The language is regular. There is a simple three-state DFA for it. I'll omit the drawing.

C. Let L match any one of the 52 letters, and let A match any one of the 62 alphanumeric characters. Then a regular expression H for hostnames is

$$(AA^*\texttt{.})^*AA^*\texttt{.}LL^*$$

Let C match any letter, digit, period, or underscore. Then a regular expression P for paths is

$$\texttt{/}(CC^*\texttt{/})^*(\epsilon \cup CC^*)$$

Let D match any of the ten digits. Then a regular expression for URLs is

$$LL^*\texttt{://}H(\epsilon \cup DD^*)(\epsilon \cup P \cup P\texttt{\#}CC^*)$$

D. Let $A = \{a^i b^j :$ exactly one of i, j is a multiple of 2, and exactly one of i, j is a multiple of 3$\} \subseteq \{a, b\}^*$. Prove that A is regular, or prove that A is not regular.

It is easy to construct DFAs that match each of these languages:

- $A_{2|i} = \{a^i b^j : 2 \text{ divides } i\}$.
- $A_{\overline{2|i}} = \{a^i b^j : 2 \text{ does not divide } i\}$.
- $A_{2|j} = \{a^i b^j : 2 \text{ divides } j\}$.
- $A_{\overline{2|j}} = \{a^i b^j : 2 \text{ does not divide } j\}$.
- $A_{3|i} = \{a^i b^j : 3 \text{ divides } i\}$.
- $A_{\overline{3|i}} = \{a^i b^j : 3 \text{ does not divide } i\}$.
- $A_{3|j} = \{a^i b^j : 3 \text{ divides } j\}$.
- $A_{\overline{3|j}} = \{a^i b^j : 3 \text{ does not divide } j\}$.

The language A is a union of intersections of these languages:

$$\begin{aligned} A \;=\; & \left(A_{2|i} \cap A_{\overline{2|j}} \cap A_{3|i} \cap A_{\overline{3|j}}\right) \cup \left(A_{2|i} \cap A_{\overline{2|j}} \cap A_{3|j} \cap A_{\overline{3|i}}\right) \\ & \cup \left(A_{2|j} \cap A_{\overline{2|i}} \cap A_{3|i} \cap A_{\overline{3|j}}\right) \cup \left(A_{2|j} \cap A_{\overline{2|i}} \cap A_{3|j} \cap A_{\overline{3|i}}\right). \end{aligned}$$

Because each of the eight languages is regular, and regular languages are closed under intersection and union, A must also be regular.

E. Let A be an infinite subset of $\{a^n b^n : n \geq 0\}$. Assume for the sake of contradiction that A is regular. By the pumping lemma, there exists a pumping length p for A. Because A is infinite, it must contain at least one string w of length at least $2p$. This $w = a^q b^q$ for some $q \geq p$. By the pumping lemma, there exist x, y, z such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^i z \in A$ for all $i \geq 0$. Clearly $y = a^k$ for some $1 \leq k \leq p$, so $xz = a^{q-k} b^q$ is not in A after all. This contradiction shows that A is not regular. Thus no infinite subset of $\{a^n b^n : n \geq 0\}$ is regular.