Define

$$CLIQUE = \{\langle G, k\rangle : G \text{ is an undirected graph, } k \geq 0\text{, and } G \text{ contains a } k\text{-clique}\}.$$

Also, for any natural number $k \geq 0$, let

$$CLIQUE_k = \{\langle G\rangle : G \text{ is an undirected graph that contains a } k\text{-clique}\}.$$

A. Show that $CLIQUE_k \in P$ for all k. (By the way, the $k = 3$ case is Problem 7.9 in our textbook.)

B. In class, we will soon learn that $CLIQUE$ is NP-complete. Without going into details, this means that if $CLIQUE \in P$, then $P = NP$. The common belief is that $P \neq NP$, and hence $CLIQUE \notin P$. Explain how it's possible that $CLIQUE_k \in P$ for all k, but $CLIQUE \notin P$.

The common belief is that NP is not closed under complementation. Explain what is wrong in each of the following "proofs" that NP is closed under complementation. (The proofs are extremely similar, but they make very different mistakes.)

C. Let $A \in NP$. Then there exists an NTM N and natural number k such that $L(N) = A$ and the running time of N is $\mathcal{O}(n^k)$. Define a TM M that, on input w, runs N on w and outputs the opposite of what N outputs. Then $L(M) = \overline{L(N)} = \bar{A}$, and the running time of M is $\mathcal{O}(n^k)$. So $\bar{A} \in NP$.

D. Let $A \in NP$. Then there exists an NTM N and natural number k such that $L(N) = A$ and the running time of N is $\mathcal{O}(n^k)$. Define an NTM M that, on input w, runs N on w and outputs the opposite of what N outputs. Then $L(M) = \overline{L(N)} = \bar{A}$, and the running time of M is $\mathcal{O}(n^k)$. So $\bar{A} \in NP$.