A. A. Let $D = 0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9$. Let

$$Z = DDDDD \cup DDDDD{-}DDDD.$$

B. Let A be a regular expression that matches all single upper-case letters, lower-case letters, and spaces ␣ . Let

$$S = DD^* ␣␣^* AA^*.$$

C. Let

$$P = \text{PO}␣␣^*\text{Box}␣␣^* DD^*.$$

D. Let C be a regular expression that matches all single upper-case letters. Let N be a regular expression that matches the newline and carriage return characters. Then the regular expression that we desire is

$$AA^* N (S \cup P) N AA^*, ␣␣^* CC ␣␣^* Z.$$

[This problem is somewhat under-specified and open-ended. In grading, I am looking for reasonable answers that demonstrate basic competence with regular expressions. In other words, a perfect answer is not required. Just about any answer can be improved to a slightly better answer that handles more obscure cases.]

B. Let A be regular and B be context-free. Let M be a DFA for A and N a PDA for B. We will design a PDA P for $A \cap B$, that simulates M and N simultaneously and accepts if and only if both M and N accept. The stack of P will be used to simulate the stack of N. Precisely, let

- $\Sigma^P = \Sigma^M = \Sigma^N$,
- $\Gamma^P = \Gamma^N$,
- $Q^P = Q^M \times Q^N$,
- $q_0^P = (q_0^M, q_0^N)$, and
- $F^P = F^M \times F^N$.

It remains to describe δ^P. For every transition $\delta^M(q^M, a) = r^M$ and $\delta^N(q^N, a, t) = (r^N, u)$, add a transition

$$\delta^P((q^M, q^N), a, t) = ((r^M, r^N), u).$$

By our usual reasoning for the product construction, P accepts exactly $A \cap B$.

C. [This is 1.49b in our textbook. By the way, 1.49a is more interesting.] Let $A = \{1^n w : n \geq 0 \text{ and } w \text{ contains at most } n \text{ 1s}\} \subseteq \{0,1\}^*$. Assume for the sake of contradiction that A is regular. Let p be the pumping length for A. Let $s = 1^p 0 1^p$. Then $s \in A$ and $|s| \geq p$. By the pumping lemma, $s = xyz$ where $y \neq \epsilon$, $|xy| \leq p$, and $xy^i z \in A$ for all $i \geq 0$. It is easy to see that xy is a substring of the first 1^p in s. Thus $y = 1^k$ for some $1 \leq k \leq p$, and $xy^0 z = 1^{p-k} 0 1^p$. When $1^{p-k} 0 1^p$ is written in the form $1^n w$, it must be true that $n \leq p - k < p$ and there are at least p 1s in w. Thus $xy^0 z \notin A$. This contradiction implies that A is not regular after all.

D. [This is 1.63a in our textbook.] Let A be infinite and regular. Because A is regular, there exists a pumping length p for A. Because A is infinite, there exists a string $s \in A$ such that $|s| \geq p$. By the pumping lemma, there exist strings x, y, z such that $y \neq \epsilon$ and $xy^i z \in A$ for all $i \geq 0$. Let $B = \{xy^i z : i \text{ is even}\}$. Because $y \neq \epsilon$, B is infinite. Because $x(yy)^* z$ is a regular expression for B, B is regular. Let $C = A - B = A \cap \bar{B}$. Because B is regular, so is $\bar{B}$. Because A and $\bar{B}$ are regular, so is their intersection, which is C. Because C contains $xy^i z$ for all odd i, C is infinite. Finally, B and C are disjoint, and $B \cup C = A$. Thus A is a disjoint union of two infinite, regular languages B and C.

E. [This is 2.9 in our textbook.] This context-free grammar works for the given language: $S \to TC|AU$, $C \to \epsilon|cC$, $A \to \epsilon|aA$, $T \to \epsilon|aTb$, $U \to \epsilon|bUc$.